Semigroups and Categories

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Workshop on Categories and semigroups
21-23 July 2022

Groupoids and semigroups

- A groupoid is a set S with respect to a binary operation.
- A groupoid S is a semigroup if the binary operation on S is associative.

Example

By a binary relation on a set X we mean a subset ρ of $X \times X$. For $a, b \in X$ and $(a, b) \in \rho$ we write $a\rho b$ and say that "a bears the relation ρ to b". If ρ and σ are relations on X then their composotion $\rho \circ \sigma$ is defined as

$$(a, b) \in \rho \circ \sigma \; \textit{if there exists} \; x \in X \; \textit{such that} \; (a, x) \in \rho, \; \textit{and} \; (x, b) \in \sigma$$

Clearly this binary operation is assocoative.
Let us denote by $B(X)$ the set of all relations on a set X. Then $B(X)$ is a semigroup under composition of relations.

Alternatevely we have the following example.

Example

Let X, be a non empty set, denote by $T(X)$ the semigroup of full transformations on X, that is, the set of all functions from X to X with function composition as the semigroup operation, we will write functions on the right and compose from left to right; that is, for $f : A \to B$ and $g : B \to C$, we will write xf , rather than $f(x)$, and $x(fg)$, rather than $(gf)(x)$. The semigroup $T(X)$ is fundamental in semigroup theory since every semigroup can be embedded in some $T(X)$

A natural generalization of $T(X)$ is the semigroup $P(X)$ of partial transformations on X (that is, functions whose domain and image are included in X). The semigroup $P(X)$ contains as its subsemigroups both $T(X)$ and the symmetric inverse semigroup $I(X)$ of partial injective transformations on X. The semigroup $I(X)$ is fundamental for the important class of inverse semigroups since every inverse semigroup can be embedded in some $I(X)$

Regular elements and inverses

Definition

An element a of a semigroup S is regular if $a \in aSa$, that is there exists an $x \in S$ such that $axa = a$. A semigroup is regular if every element of S is regular.

Two elements a and b in a semigroup S are said to be inverses to each other if

$$aba = a \quad \text{and} \quad bab = b.$$

Note that in a regular semigroup every element has inverse, however the inveses need not be unique.

Definition

An inverse semigroup is a semigroup in which every element had a unique inverse.

Idempotents and Band semigroups

An element $a \in S$ is an idempotent if $e \cdot e = e$ and the set of all idempotent elements in S is denoted by $E(S)$. A semigroup in which every element is an idempotent is a band semigroup.

Example

(Rectangular Band) Let X and Y be any two sets. Define a binary operation on $S = X \times Y$ as follows

$$(x_1, y_1)(x_2, y_2) = (x_1, y_2) \quad x_1, x_2 \in X \, and \, y_1, y_2 \in Y.$$

It is easy to see that S is a band semigroup and is called rectangular band.

Note the following

- Every regular element in a semigroup S has at least one inverse.
- The semigroup of all partial injections on a set is an inverse semigroup.
- Inverse semigroups are regular.
- Rectangular band is a regular semigroup which is not an inverse semigroup.

Ideals

Definition

A subset I of a semigroup S is called a left ideal of S if $sa \in I$ for all $a \in I$ and $s \in S$.

The right and two sided ideals may defined analogously.

1. For $A \subseteq S$, the left ideal of S generated by A is $A \cup SA$ and it is the smallest left ideal containing A.
2. If $A = \{a\}$ the ideal generated by A is called the principal left ideal generated by a and it is denoted by Sa.
3. Analogously one can define right ideal and two sided ideals generated by a subset as well as an element of the semigroup.

Congruences and factors

A relation ρ on a semigroup S is said to be left [right] compatible if $a\rho b \ (a, b \in S)$ imples $ac\rho bc\,[ca\rho cb]\,]$ for every $c \in S$. A left [right] compatible equivalence relation on S is called a left [right] congruence. If ρ is a congruence on a semigroup S, then ρ produces a factor S/ρ which is again a semigroup and is called the quotiet (factor) semigroup.

Example

Let I be an ideal in a semigroup S. For $a, b \in S$, define $a\rho b$ when $a = b$ or else a and b belongs to the ideal I. Then ρ is a congruence and is called the Rees congruence.

Simple and 0-simple semigroups

1. The semigroup S is called left simple if $Sa = S$ for every $a \in S$. ie., S admits no proper left ideals. Similarly right simple and simple semigroups if it does not have any proper right ideal or proper ideal respectively.
2. If S contains 0, $\{0\}$ is an ideal. S is called 0-simple when $\{0\}$ is the only proper ideal of S.

Example

$(G, \cdot)$ be positive rationals with $\cdot$ the usual multiplication and let $A = G \times G$. Define a product on A by

$$(a, b)(c, d) = (ac, bc + d) \quad a, b, c, d \in G$$

then A is a simple semigroup.

Green's relations

Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{D}$ and $\mathcal{H}$ are equivalence relations on S defined as,

1. $a\mathcal{L}b$ if and only if $S^1a = S^1b$, i.e, if and only if a and b generate the same principal left ideal.
2. $a\mathcal{R}b$ if and only if $aS^1 = bS^1$, i.e, if and only if a and b generate the same principal right ideal.
3. $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$, i.e, if and only if a and b generate the same principal ideal.
4. $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, i.e, the smallest equivalence relation containing both $\mathcal{L}$ and $\mathcal{R}$.
5. $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, i.e, the largest equivalence relation contained in both $\mathcal{L}$ and $\mathcal{R}$.

Clearly $\mathcal{H} \subseteq \mathcal{L}, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$. The equivalence classes of these relations are called $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{D}$ and $\mathcal{H}$-classes respectively.

Simple and bisimple semigroups

- A semigroup is left simple if and only if it consists of a single $\mathcal{L}$-class.
- A semigroup is right simple if and only if it consists of a single $\mathcal{R}$-class.
- A semigroup is simple if and only if it consists of a single $\mathcal{J}$-class.
- A semigroup is S is bisimple (or $\mathcal{D}$-simple) if it consists of a single $\mathcal{D}$ class.
- Rectangular band is a bisimple semigroup.

Lemma

Let S be a semigroup with 0 such that 0 is the only proper two sided ideal of S . then either S is 0-simple or S is a null semigroup of order 2.

Regular $\mathcal{D}$- classes

Theorem

1. If a $\mathcal{D}$-class D of a semigroup S contains a regular element, then every element of D is regular.
2. If $\mathcal{D}$ is regular, then every $\mathcal{L}$-class and $\mathcal{R}$-class contained in D contains an idempotent.

It is also important to note the following

> An element a in a semigroup S is regular if and only if the principal left ideal generated by a has an idempotent generator and the principal right ideal generated by a has an idempotent generator.

Inverse semigroups

Theorem

The following conditions on a semigroup S are equivalent.

1. S is regular and any two idempotents of S commutes with each other.
2. Every principal left[right] ideals of S has unique idempotent generator.
3. S is an inverse semigroup.

The Wagner-Preston representation theorem

Let S be an inverse semigroup. Then there is a set X and an injective homomorphism $\theta : S \to I_X$.

That is, every inverse semigroup is isomorphic with a subsemigroup of an appropriate symmetric inverse semigroup.

Semisimple semigroups

Let $\mathcal{J}(a)$ denote the principal ideal $S^1 a S^1$ generated by a in S and $\mathcal{J}_a$ be the $\mathcal{J}$-class containing a. ie., the set of generaters of $\mathcal{J}(a)$.
Let $I(a) = \mathcal{J}(a) - \mathcal{J}_a$.
If $I(a)$ non empty, it is an ideal of S, in particular it is an ideal of $\mathcal{J}(a)$. The Rees factor semigroup $\mathcal{J}(a)/I(a)$ with $a \in S$ is called the principal factor of S.

Definition

A semigroup is called semisimple if every principal factor of S is simple or 0-simple or simple.

Theorem

Every ideal of an ideal of a semisimple semigroup S is an ideal of S.

Completely 0-simple semigroups

Primitive idempotent in a semigroup is the minimal element in the set of idempotents under the natural partial order.
If S is a semigroup with zero, a primitive idempotent is the minimal non-zero idempotent under the natural partial order.

Definition

A semigroup is completely 0-simple if it is 0-simple and contains a primitive idempotent.

The following theorem is due to Green and is interesting to write a proof.

Theorem

A completely 0-simple semigroup is 0-bisimple and regular.

Rees matrix semigroups I

1. A Rees matrix semigroup $S = \mathcal{M}^0(G; I, \Lambda; P)$ is a semigroup whose elements are $I \times \Lambda$ matrices over $G \cup \{0\}$ having atmost one nonzero entry.
2. The binary operation is sandwich matrix multiplication with respect to $P = [p_{\lambda i}]_{\Lambda \times I}$. The matrix $P = [p_{\lambda i}]_{\Lambda \times I}$ is called the sandwich matrix.
3. A matrix having its unique nonzero entry $g \in G$ at the $(i, \lambda)^{th}$ position can be identified as the triplet (g, i, λ).
4. The binary operation is given by,

$$(g, i, \lambda)(h, j, \mu) = \begin{cases} (g p_{\lambda j} h, i, \mu) & \text{if } p_{\lambda j} \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Rees matrix semigroups II

5. The matrix P is said to be regular if each row and each column of P has atleast one nonzero entry.
6. A regular Rees matrix semigroup is a Rees matrix semigroup with regular sandwich matrix.

Rees theorem

A semigroup is completely 0-simple if and only if it is isomorphic with a regular Rees matrix semigroup over a group.

Generalized Green's relations and Abundant semigroup

Inorder to extend the study of the structure theory beyond regular semigroups, it will be desirable to extend the Greens relation. The Generalized Greens relations are defined presicely with this motivation. This idea was first introduced by McAlister and Pastjin.

The Generalized Greens relations $\mathcal{L}^*$, $\mathcal{R}^*$, $\mathcal{D}^*$, $\cdots$ are defined as follows

1. $a\mathcal{R}^*b$ if and only if $a\mathcal{R}b$ in some semigroup $\bar{S}$ containing S.
2. $a\mathcal{L}^*b$ if and only if $a\mathcal{L}b$ in some semigroup $\bar{S}$ containing S.
3. The join of $\mathcal{R}^*$ and $\mathcal{L}^*$ be $\mathcal{J}^*$.
4. $\mathcal{H}^* = \mathcal{R}^* \cap \mathcal{L}^*$

and the composite of $\mathcal{L}^*$ and $\mathcal{R}^*$ is $\mathcal{D}^*$.

Definition

A semigroup is abundant if every $\mathcal{R}^*$- class and $\mathcal{L}^*$-class contains an idempotent.

Abundant semigroup

Lemma

The following are equivalent for elements $a, b \in S$:

1. $a\mathcal{R}^*b$
2. for all $x, y \in S^1,\ xa = ya$ if and only if $xb = yb$.

This condition is simplified when one of the elements is an idempotent.

Corollary

Let a be an element of a semigroup S, and let $e \in E(S)$. Then the following are equivalent:

1. $a\mathcal{R}^*e$
2. $ea = a$ and for all $x, y \in S^1,\ xa = ya$ implies $xe = ye$.

Categories

K.S.S. Nambooripad brilliantly used category theory to describe the structure of regular semigroups in the theory of Cross-connections. In this regard, he considered the ideal categories of semigroups, these categories are *categories with subobjects and factorization*

Example

1. *A category $\mathcal{C}$ whose objects are semigroups and morphisms homomorphisms is the category of semigroups and will be denoted as* **Sgrp**.
2. *The category* **Sgrp** *has a natural subobject relation.*
3. *Every morphisms admits a factorization.*

Subobjects and Factorization

If there exists a choice of subobjects P in the category $\mathcal{C}$, then the pair $(\mathcal{C}, P)$ is called a category with subobjects .
A category $\mathcal{C}$ is said to have factorization properity if every $f \in \mathcal{C}$ can be expressed as $f = pm$ where p is an epimorpism and m is an embedding. A factorization of the form $f = qj$ where q is an epimorphism and j is an inclusion is called a canonnical factorization. A category $\mathcal{C}$ is said to have factorization property if and only if every morphism in $\mathcal{C}$ admits a cannonical factorization. A factorization of a morphism f of the form $f = euj$ where e is a retraction, u is anm isomorphism and j is an inclusion is called a normal factorization of f.

Definition

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ is an ideal of $\mathcal{C}$ if $\mathcal{C}'$ is a full subcategory of $\mathcal{C}$ such that $\langle c \rangle \subseteq \mathcal{C}'$ for all $c \in v\mathcal{C}'$. The ideal $\langle c \rangle_{\mathcal{C}}$ is called the ideal generated by c

Cones in categories

Definition

Let $\mathcal{C}$ be a category with subobjects and $d \in v\mathcal{C}$. A map $\gamma : v\mathcal{C} \to \mathcal{C}$ is called a cone from the base $v\mathcal{C}$ to the vertex d if

1. $\gamma(c) \in \mathcal{C}(c, d)$ *for all* $c \in v\mathcal{C}$
2. *if* $c \subseteq c'$ *then* $j^{c}_{c'}\gamma(c') = \gamma(c)$

For a cone γ denote by c_γ the vertex of γ and for $c \in v\mathcal{C}$, the morphism $\gamma(c) : c \to c_\gamma$ is called the component of γ at c. A cone γ is said to be normal if there exists $c \in v\mathcal{C}$ such that $\gamma(c) : c \to c_\gamma$ is an isomorphism. We denote by $T\mathcal{C}$ the set of all normal cones in $\mathcal{C}$ and by M_γ, the set

$$M_\gamma = \{c \in v\mathcal{C} : \gamma(c) \text{ is an isomorphism } \}.$$

Normal Categories

Definition

A category $\mathcal{C}$ with subobjects is called a normal category if the following holds

1. *any morphism in $\mathcal{C}$ has a normal factorization*
2. *every inclusion in $\mathcal{C}$ splits*
3. *for each $c \in v\mathcal{C}$ there is a normal cone γ with vertex c and $\gamma(c) = 1_{c_\gamma}$.*

Observe that given a normal cone γ and an epimorphism $f : c_\gamma \to d$ the map $\gamma * f : a \to \gamma(a)f$ from $v\mathcal{C}$ to $\mathcal{C}$ is a normal cone with vertex d. Consider two normal cones γ and σ, then

$$\gamma \cdot \sigma = \gamma * (\sigma(c_\gamma))^\circ$$

where $(\sigma(c_\gamma))^\circ$ is the epimorphic part of $\sigma(c_\gamma)$, defines a binary composition on $T\mathcal{C}$.

Semigroup of normal cones

Theorem

Let $\mathcal{C}$ be a normal category. Then $T\mathcal{C}$ the set of all normal cones in $\mathcal{C}$ is a regular semigroup with the binary operation

$$\gamma \cdot \sigma = \gamma * (\sigma(c_\gamma))^\circ \tag{1}$$

and $\gamma \in T\mathcal{C}$ is idempotent if and only if $\gamma(c_\gamma) = 1_{c_\gamma}$.

Regular semigroup and normal categories

For a regular semigroup S every principal left(right) ideal is generated by an idempotent. In [4] it is shown that the category $\mathbb{L}(S)\,[\mathbb{R}(S)]$ whose objects are principal left left ideals $\{Se : e \in E(S)\}$ [right ideals $\{eS : e \in E(S)\}$] of a regular semigroup S and morphisms any morphism from Se to Sf in $\mathscr{L}(S)$ is a partial right translation $\rho(e, u, f) : u \in eSf$. [$eS$ to fS in $\mathscr{R}(S)$ is a partial left translation] is a normal categoriy.

Abundant semigroup Proper categories of

In order to extend the study of the structure theory of regular semigroup via categories to non regular semigroups, in [5] P.G.Romeo consider a category $\mathcal{C}$ with sub objects in which inclusions splits and every morphism has canonical factorization. A factorization of a morphism f of the form euj where e is a retrction, u is a balanced morphism (ie., a morphism which is both monic and epi) and j an inclusion is termed as a balanced factorization.

Definition

A cone γ with vertex d in $\mathcal{C}$ such that there exists at least one $c \in v\mathcal{C}$ with $\gamma(c) : c \to d$ is a balanced morphism is a proper cones in $\mathcal{C}$ and we denote the set of all cones in $\mathcal{C}$ by $\mathcal{PC}$.

Definition

A small category $\mathcal{C}$ with sub objects is called proper category if it satisfies the following:

1. *every inclusion in $\mathcal{C}$ splits,*
2. *every morphism $f \in \mathcal{C}$ has unique canonical factorization and*
3. *for each $a \in v\mathcal{C}$, there exists $\gamma \in \mathcal{PC}$ such that $\gamma(a) = I_a$.*

Theorem

The set of all proper cones $\mathcal{PC}$ in the category $\mathcal{C}$ is a semigroup with respect to the binary operation defined

$$\gamma \cdot \eta = \gamma \star \eta(c_\gamma)^o$$

where $\gamma, \eta \in P\mathcal{C}$.

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Problems I

1. Prove that the following conditions on a semigroup S are equivalent.
    1. S is regular and any two idempotents of S commutes with each other.
    2. Every principal left[right] ideals of S has unique idempotent generator.
    3. S is an inverse semigroup.
2. Every principal left ideals of a left zero semigroup with more than one element has a unique idempotent generator. But it is not an inverse semigroup. What are the reasons for that?
3. In an inverse semigroup every element has a unique inverse. So every idempotent is the inverse of itself. Give an example of a semigroup and an idempotent having inverse other than itself.
4. Let S be a semigroup with 0. Then prove that, S is both left 0-simple and right 0-simple if and only if it is a group with zero.

Problems II

5. Let S be a regular semigroup with zero and $e \in E(S)$ be a primitive idempotent. Then eS is 0-minimal right ideal and Se is 0-minimal left ideal.
6. Identify the Green's relations for the full transformation semigroup.
7. Prove that every finite 0-simple semigroups are completely 0-simple.

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THANK YOU