IMRT WORKSHOP ON CATEGORIES AND SEMIGROUPS

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Introduction

We begin with two quotes.
Tom Leinster author of the book
Basic Category Theory (Cambridge 2014)
begins his introduction as follows.

> *Category theory takes a birds eye view of Mathematics.*
> *From high in the sky details become invisible,*
> *but we can spot patterns that were impossible to detect*
> *from ground level.*

Another quote is from
Emily Riehl's *Category Theory in Context* (Dover Modern Maths. 2016)

> *Category theory provides a cross disciplinary language for Mathematics designed to delineate general phenomena which enables the transfer of ideas from one area of studies to another.*

A third view which we use currently is to regard categories as an alge braic structure which generalises
groups
groupoids
semigroups
etc.

The concept of categories was introduced by Mac Lane and Eilenberg in 1940's as framework to deal with classes of algebraic structures such as the class of groups, class of rings, vector spaces, linear and topological spaces etc.

The theory could provide global descriptions of various properties in these structures. For example the concepts of free objects, quotient objects, direct products etc. could be given common description.

Theorems on categories could thus provide specific theorems on each of the classes considered.
The basic idea behind this generalization is to translate results on each structure into results on structure preserving mappings.

The term *homomorphism* is generalised to *morphism* in categories. Thus if we can state results on an algebraic structure in terms of the associated morphisms then it could be made valid in other structures as well.

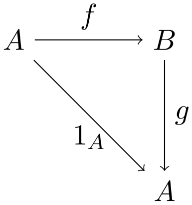
For example a map $f : A \to B$ between sets A and B is a one to one map if for $a, b \in A$

$$f(a) = f(b) \Rightarrow a = b.$$

A translation of this in terms of mappings is as *right invertible morphism* which is defined as follows.
$f : A \to B$ is said to be right invertible if there exists $g : B \to A$ such that

$$fg = 1_A$$

where 1_A is the identity map on A.


A
f
B
g
1A
A

Similarly onto mappings may be related to left inveribility and *Isomorphisms* can be seen as *Invertible morphisms*.
For example in the category of topological spaces if consider continuous maps as morphisms we can see that an isomorphism is a continuous maps having a continuous inverse. Here isomorphisms are usually called *Homeomorphisms*.

The role of categories as algebraic structures
is obtained by considering the morphisms as the
significant part of a category.
In small categories this is a set with a *partial binary operation*
where the operation is the composition of morphisms.
This way a category can be realised as a generalization of
semigroups.

The concept of normal category introduced by KSS Nambooripad and several related categories uses this aspect very effectively.

In this introductory talk we discuss the following concepts.

- Categories and Morphisms
- Functors
- Hom functors
- Natural Transformations
- Natural Isomorphisms

Categories and Morphisms

A category $\mathcal{C}$ is a structure with two components: objects and morphisms.
All the objects of $\mathcal{C}$ form a class denoted by $v\mathcal{C}$. When $v\mathcal{C}$ is a set in the sense of set theory we call $\mathcal{C}$ a *small category*. While considering a category as an algebraic generalization of semigroups we restrict to small categories.
So in the followig we consider ony small categories.

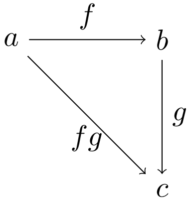
Definition 1

A category (to be more precise, a small category) is a pair $(v\mathcal{C},\ m\mathcal{C})$ where $v\mathcal{C}$ is called the set of objects and $m\mathcal{C}$ is called the set of morphisms satisfying the following.

- *With each morphism f in $\mathcal{C}$ there is associated two objects a, called the domain of f, and b, called the codomain of f. We denote this relation by writing $f : a \to b$. We denote the set of all morphisms with domain a and codomain b as*

$$\mathcal{C}(a, b) \text{ or } [a, b]_{\mathcal{C}} \text{ or } [a, b] \text{ or } hom(a, b).$$

- There is a composition $\mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c)$ denoted by fg for $f : a \to b$ and $g : b \to c$



- such that
$$f(gh) = (fg)h$$
whenever the products are defined.
- Every morphism $f : a \to b$ has a left identity 1_a and a right identity 1_b such that
$$1_a f = f \text{ and } f 1_b = f.$$

We usually denote a category $(v\mathcal{C}\ , m\mathcal{C})$ by $\mathcal{C}$ only. Also we write $\mathcal{C}$ to denote the morphism class $m\mathcal{C}$ so that $f \in \mathcal{C}$ means that f is a morphism in $\mathcal{C}$.

Examples

A category with sets as objects and mappings between sets as morphisms is the notation setting example of a category. It is denoted by $\mathcal{S}et$.

Note that $f : A \to B,\ 1_A, fg$ etc. have same meaning in set theory and category theory.

Similarly one can consider categories of
groups with homomorphisms as morphisms,
topological spaces with continuous maps as morphisms,
lattices with meet and join preserving mappings as morphisms etc.

Some different types of examples are the following.
1. Let X be any nonempty set and $T(X)$ be the category with only one object X and all mappings from X to X as morphisms. Here we may denote the set of all morphisms also by $T(X)$. Then clearly $T(X)$ is a semigroup under the composition of maps as the semigroup operation. Moreover it is a monoid.

Instead of considering all mappings from X to X as morphisms if we consider only bijections as morphisms. The resulting category is then a group.
We may also consider several specialised categories also in this context.

2. Category with One object.
Let $\mathcal{C}$ be a category with one object. Let $\mathcal{C}$ also denote the set of morphisms.
Then $\mathcal{C}$ is a monoid.
Conversely every monoid S can be considered as a category with one object.
Thus any category with only one object is a monoid.

3. Category with two objects.
Let $\mathcal{C}$ be a category with $v\mathcal{C} = \{a, b\}$.
Then the category contains two monoids $\mathcal{C}(a, a)$ and $\mathcal{C}(b, b)$.

4. A category in which all morphisms are isomorphisms is called a groupoid.
If a category $\mathcal{C}$ with $v\mathcal{C} = \{a, b\}$ is a groupoid then the category contains two groups $\mathcal{C}(a, a)$ and $\mathcal{C}(b, b)$.
If further $\mathcal{C}(a, b)$ and $\mathcal{C}(b, a)$ are empty then this category is simply a union of two groups.

5. Another category that appears often in applications is a category related to partial orders and quasi orders.
A *preorder* is a category in which for any pair of objects a, b the morphism set $[a, b]$ is either empty or singleton.
If in addition $[a, b] \cap [b, a] = \emptyset$ for any pair of objects a, b with $a \neq b$ then the category is called a partial order.

These categories can be realised as alternate descriptions of quasi ordered sets and partially ordered sets.
We define a relation $\leq$ on set of objects of a small category by

$$a \leq b \text{ if } [a, b] \neq \emptyset.$$

If $\mathcal{C}$ is a preorder then $\leq$ is a quasi order on $v\mathcal{C}$.
If $\mathcal{C}$ is a partial order then $\leq$ is a partial order on $v\mathcal{C}$.

Conversely if $(X, \leq)$ is a quasi ordered set then X can be realised as a category in which the elements of X are objects and for $a, b \in X$ the morphism set $[a, b]$ either empty or singleton and is nonempty if and only if $a \leq b$ in X.
In this case we may denote the morphism from a to b by the pair (a, b) whenever $a \leq b$ so that

$$(a, b)(b, c) = (a, c)$$

whenever $a \leq b$ and $b \leq c$.
Thus X is a category which is a preorder.

Functors and Natural Transformations

These are two associated transformations on categories.
Functors are structure preserving mappings of categories and Natural Transformations are relations between functors.

The following is the definition.

Definition 2

Let $\mathcal{C}, \mathcal{D}$ be categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$ is a pair of mappings, one from $v\mathcal{C}$ to $v\mathcal{D}$ and the other from $m\mathcal{C}$ to $m\mathcal{D}$ (both denoted by F for convenience) satisfying the following.

- If $f : a \to b$ in $\mathcal{C}$ then $F(f) : F(a) \to F(b)$ in $\mathcal{D}$.
- For each $a \in v\mathcal{C}$
$$F(1_a) = 1_{F(a)}.$$
- For $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$
$$F(fg) = F(f)F(g).$$

When categories in which objects are again categories is considered often the morphisms are taken as functors.
Moreover *isomorphism* between categories is defined as a functor which is invertible.
That is a functor $F : \mathcal{C} \to \mathcal{D}$ is an isomorphism of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that

$$FG = 1_{\mathcal{C}} \text{ and } GF = 1_{\mathcal{D}}.$$

Here composition of functors arise as composition of the mappings on objects and on mprphisms in the categories.

Several general constructions in algebraic structures can be realised as functor.
For example the free group construction can be realised as a functor F from the category of sets to the category of groups by setting

$$F(X) = \text{ the free group on } X.$$

Similarly the assignment $X \mapsto (X, \tau_d)$ is a functor from the category of sets to the category of topological spaces where τ_d denotes the discrete topology on X.
The morphisms in the category of topological spaces are continuous maps.
Also $(X, \tau) \mapsto (X, \tau_d)$ can be seen to be a functor from the category of topological spaces into itself.

Natural transformations are relations between functors described as follows.

Definition 3

Let $\mathcal{C}, \mathcal{D}$ be categories and F, G be functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation $\eta : F \to G$ is a collection $\{\eta_a : a \in v\mathcal{C}\}$ of morphisms in $\mathcal{D}$ such that the following hold.

- For each $a \in v\mathcal{C}, \eta_a$ is from $F(a)$ to $G(a)$.
- For $f : a \to b$ in $\mathcal{C}$

$$\eta_a G(f) = F(f)\eta_b.$$

That is the following diagram is commutative.

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\eta_b]{} & G(b) \end{array}$$

Example 4

Examples of natural transformations can be found in several general constructions in algebra. For example consider the abelianization of groups.
Given a group G the quotient $G/[G, G]$ where $[G, G]$ is the commutator subgroup of G is the maximum abelian quotient of G.
Let $F : \mathcal{G}rp \to \mathcal{G}rp$ be the functor which maps every group G to the abelian quotient $G/[G, G]$. Let $I : \mathcal{G}rp \to \mathcal{G}rp$ be the identity functor.
Then

$$\eta : I \to F \text{ with } \eta_G : G \to G/[G, G]$$

as the quotient homomorphism is a natural transformation.

That is the following diagram is commutative.

$$\begin{array}{ccc} G & \xrightarrow{\eta_G} & F(G) = G/[G,G] \\ f \downarrow & & \downarrow F(f) \\ H & \xrightarrow[\eta_H]{} & F(H) = H/[H,H] \end{array}$$

Example 5

Another example is given by the free group construction.
Let $F : \mathcal{S}et \to \mathcal{G}rp$ be the functor which maps every set X to the free group $F(X)$ on X. We consider $UF(X)$ as the underlying set of the group $F(X)$. Let $I : \mathcal{S}et \to \mathcal{S}et$ be the identity functor.
For each set X let

$$\eta_X : X \to UF(X)$$

be the inclusion map.
Then $\eta : I \to UF$ is a natural transformation.

That is the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & UF(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle UF(f)} \\ Y & \xrightarrow[\eta_Y]{} & UF(Y) \end{array}$$

Example 6

Another class of examples of functors and natural transformations that occur frequently in discussions on categories is the following. Let $\mathcal{C}$ *be a category.*
For each object a *in* $\mathcal{C}$ *a functor* $Hom(a, -) : \mathcal{C} \to \mathcal{S}et$ *is defined as follows. For* $c, d \in v\mathcal{C}$ *and* $f : c \to d$

$$Hom(a, -)(c) = Hom(a, c) = [a, c]_{\mathcal{C}}$$

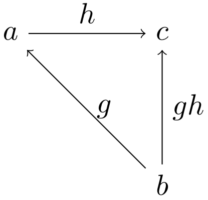
and $Hom(a, -)(f) = Hom(a, f) : Hom(a, c) \to Hom(a, d)$ *is defined by*

$$h \mapsto hf$$

for all $h : a \to c$*. These functors are called homfunctors.*

There is a naturally defined natural transformation between these homfunctors. Let $Hom(a, -)$ and $Hom(b, -)$ be homfunctors and $g : b \to a$ be a morphism. Then $Hom(g, -) : Hom(a, -) \to Hom(b, -)$ is a natural transformation defined by

$Hom(g, -)_c = Hom(g, c) : Hom(a, c) \to Hom(b, c)$ mapping $h \mapsto gh$.



It is easy to see that every natural transformation from $Hom(a, -)$ to $Hom(b, -)$ is determined by a morphism $g : b \to a$.

Theorem 7

Let $\eta : Hom(a, -) \to Hom(b, -)$ be a natural transformation. Then there exists a morphism $g : b \to a$ such that $\eta = Hom(g, -)$.

Proof.

Consider the following commutative diagram given by the natural transformation.

$$\begin{array}{ccc} Hom(a,a) & \xrightarrow{\eta_a} & Hom(b,a) \\ {\scriptstyle Hom(a,f)}\downarrow & & \downarrow{\scriptstyle Hom(b,f)} \\ Hom(a,c) & \xrightarrow[\eta_c]{} & Hom(b,c) \end{array}$$

□

Taking $g = (1_a)\eta_a \in Hom(b, a)$ we see that

$$(f)\eta_c = gf$$

for all $f \in Hom(a, c)$. This gives the following commutative diagram.

$$\begin{array}{ccc} Hom(a,a) & \xrightarrow{Hom(g,a)} & Hom(b,a) \\ {\scriptstyle Hom(a,f)}\downarrow & & \downarrow{\scriptstyle Hom(b,f)} \\ Hom(a,c) & \xrightarrow[Hom(g,c)]{} & Hom(b,c) \end{array}$$

The following terminology relating to special morphisms in a category is commonly used.

Definition 8 (Monomorphism, Epimorphism, and Isomorphism)

Let $\mathcal{C}$ be a category and $f \in \mathcal{C}$.

- *f is said to be a monomorphism if $gf = hf$ implies $g = h$ for any morphisms $g, h \in \mathcal{C}$.*
- *f is said to be an epimorphism if $fg = fh$ implies $g = h$ for any morphisms $g, h \in \mathcal{C}$.*
- *An isomorphism is a morphism which is both left and right invertible. That is $f : a \to b$ is an isomorphism if and only if there is $g : b \to a$ such that*

$$fg = 1_a \text{ and } gf = 1_b.$$

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