GENERALIZED INVERSES IN SEMIGROUP OF STOCHASTIC MATRICES

A R Rajan
Director IMRT
Former Professor and Head
Department of Mathematics, University of Kerala
Email-arrunivker@yahoo.com website: imrt.org.in

Introduction

This is an exposition on the paper of Wall (cf. [8]) on generalized inverses of stochastic matrices.
Wall characterises all stochastic matrices admitting a stochastic matrix as its generalised inverse.
He provides a method for computation of these generalised inverses as well.
Also given is a characterization of stochastic matrices admitting stochastic Moore-Penrose inverse.

Generalized Inverses

We begin with familiarization of the terminology and basic concepts.

Generalized inverses are termed as *semi-inverses* in the paper of Wall.

For a matrix A a generalised inverse is a matrix X such that

$$AXA = A \text{ and } XAX = X$$

The Moore-Penrose inverse of A is a generalized inverse X such that

$$(AX)^* = AX \text{ and } (XA)^* = XA$$

where $*$ denotes adjoint.
That is AX and XA are self adjoint projections.
We denote the Moore-Penrose inverse by A^+.

It is known that every matrix has a Moore-Penrose inverse and it is unique.
Well defined formula for computation of Moore-Penrose inverse using a full rank factorization of the matrix is avilable in literature. For example the book [4].

The following is the procedure given in [4]. Let A be an $n \times n$ matrix of rank k.
Let
$$A = FG$$
be a full rank factorization of A.
Then F is an $n \times k$ matrix of rank k and
G is $k \times n$ matrix of rank k.

Let

$$F^+ = (F^*F)^{-1}F^*$$

and

$$G^+ = G^*(GG^*)^{-1}.$$

Then the Moore-Penrose inverse A^+ is

$$A^+ = G^+F^+.$$

In Linear Algebra contexts the matrix A can be any $m \times n$ matrix. In Semigroup Theory we confine to $n \times n$ square matrices so that they form a semigroup.

Stochastic Matrices

An $m \times n$ matrix $A = (a_{ij})$ is said to be a **stochastic** matrix if all its entries are non negative and

$$\sum_j a_{ij} = 1$$

for all i.
This may be called row stochastic.
In a similar way column stochastic matrices can also be defined.
Combining the two we have **doubly** stochastic matrices also.
It may be observed that any doubly stochastic matrix is a square matrix.

Theorem 1

The set SM_n of all stochastic $n \times n$ matrices is a semigroup.

Proof.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be stochastic $n \times n$ matrices. Then

$$\sum_j a_{ij} = 1 \text{ and } \sum_j b_{ij} = 1$$

for all i.
Let $AB = (c_{ij})$ so that

$$c_{ij} = \sum_k a_{ik}.$$

□

Then

$$\begin{aligned}\sum_j c_{ij} &= \sum_j (\sum_k a_{ik} b_{kj}) \\ &= \sum_k a_{ik} (\sum_j b_{kj}) \\ &= \sum_k a_{ik} \\ &= 1.\end{aligned}$$

For doubly stochastic matrices the following is a known theorem.

Theorem 2 (Plemmons and Cline)

Let A be a doubly stochastic matrix. If A has doubly stochastic generalized inverse then it is unique and is A^T. Furter in this case $A^+ = A^T$ where A^+ is the Moore-Penrose inverse of A.

The following example shows that a stochastic matrix may not have a stochastic gneralized inverse.

Example 1

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}.$$

It is doubly stochastic and has no doubly stochastic generalised inverse since A^T is not a generalized inverse of A.

Here A is invertible and its inverse is

$$A^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

So A has no stochastic generalized inverse also.

To consider a non invertible case we may modify the above matrix as follows.

$$B = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}.$$

Now B is a 4×4 matrix of rank 3.
It can be seen that any generalized inverse of B is of the form

$$B' = \begin{pmatrix} X & D \\ E & Y \end{pmatrix}$$

so that X is a generalized inverse of A.

So

$$X = A^{-1} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}.$$

Thus we see that B' can not be stochastic.

In the article [8], Wall characterizes Stochastic matrices which admit stochastic generalized inverse and gives a method of constructing such generalised inverses.

Two main theorems of the above article are the following.

- Characterization of stochastic matrices admitting stochastic generalized inverses.
- Characterization of stochastic matrices for which the Moore-Penrose inverse is stochastic.
- Method of constructing the generalised inverses in case they exist.

It may be observed that all stochastic matrices admitting stochastic generalized inverse may not have stochastic Moore-Penrose inverse.

$$A = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

It is a stochastic matrix and has stochastic generalised inverse. But its Moore-Penrose inverse is not stochastic.

Here

$$A^{+} = \begin{pmatrix} \frac{3}{10} & \frac{3}{10} \\ \frac{3}{5} & \frac{3}{5} \end{pmatrix}.$$

Here A is an idempotent matrix and so A is a generalized inverse of A.
Thus A has a stochastic generalized inverse.

Also

$$A' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is another generalised inverse of A and A' is stochastic.

$$A'' = \begin{pmatrix} 1 & 1 \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

is another generalized inverse of A and this is not stochastic.

The following theorem is relevant here.

Theorem 3 (Swarz)

Let E be an idempotent $n \times n$ stochastic matrix of rank k. Then the maximal subgroup H_E in SM_n containing E is isomorphic to the symmetric group S_k.

An immediate corollary is the following.

Theorem 4

Let E be an idempotent $n \times n$ stochastic matrix of rank 1. Then the maximal subgroup H_E containing E is the trivial group $\{E\}$.

The following theorem appears in [9].

Theorem 5 (Schwarz)

Idempotent rank one stochastic matrices form a right zero semigroup.

It may also be noted that all stochastic rank 1 matrices are idempotents.
Also it can be observed that all stochastic rank 1 matrices have repeated rows.

In the case of doubly stochastic matrices (ie. both row stochastic and column stochastic) there is only one rank 1 idempotent $n \times n$ matrix. This is

$$\begin{pmatrix} 1/n & 1/n & \cdots & 1/n \\ 1/n & 1/n & \cdots & 1/n \\ & \vdots & \vdots & \\ 1/n & 1/n & \cdots & 1/n \end{pmatrix}$$

Any 2×2 stochastic rank 1 matrix is of the form

$$\begin{pmatrix} \lambda & 1-\lambda \\ \lambda & 1-\lambda \end{pmatrix}$$

for $0 \leq \lambda \leq 1$.

Qn. Is the maximal subgroup H_E in SM_n equal to the maximal subgroup H_E in M_n.

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Now some related questions are the following.

Qn. Compare Green's relations in M_n and SM_n.
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are they $\mathcal{D}$-related in SM_n.
Premchand thesis gives some answers.

Qn. Does the set of all regular elements of SM_n form a subsemigroup of SM_n.

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