REGULARITY OF THE SEMI-GROUP OF REGULAR PROBABILITY MEASURES ON COMPACT HAUSDORFF TOPOLOGICAL GROUPS

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Abstract. There are many deep results on the structure of *REGULAR* probability measures $P(G)$ on compact/locally compact, Hausdorff topological groups G. See, for instance, the classic monographs by KR Parthasarathy [15], Ulf Grenander [6] A.Mukherjea and Nicolas A.Tserpes [13]. It is known that the set $P(G)$ forms a semi-group under convolution.

Wendel in his remarkable paper [16] proved a basic result regarding support of convolution of two probability measures. Consequently, he established that the semi-group $P(G)$ is not a group. In this short paper, it is proved that for a compact topological group G, the semi-group $P(G)$ of probability measures is not **algebraically regular**. However, there are concrete regular semi-groups in which $P(G)$ can be embedded.

1. Introduction

As mentioned in the abstract, it is well known that the set $P(G)$ of $REGULAR$ probability measures on a topological group G is a semi-group under convolution, which is abelian if and only if the group G is abelian. It is also known that $P(G)$ is a compact convex set under the weak* topology of measures. Wendel [16] established many significant results regarding the algebraic, topological as well as geometric structure of $P(G)$. He showed that $P(G)$ is a closed convex semi-group which is not a group except for trivial groups G by showing that the only invertible elements are point mass measures supported on single elements.

The problem we consider is the **regularity** of $P(G)$. A semigroup is called **regular** if each of its element has a genealised inverse.

The main theorem proved in this article is Theorem 3.3, which states that $P(G)$ is not a regular semi-group unless, of course, for the trivial case $G = \{e\}$. In section 4, the embedding of this semi-group into a regular one is considered. In the concluding section 5, several related problems are given, such as the optimality of this embedding. However, a possible groundwork is prepared using the already existing theory of non-commutative Fourier transform of measures in $P(G)$ for the special case where G is a compact Lie group [1].

2. Priliminaries

Let G be a compact, Hausdorff topological group and $\mathcal{B}$ denote the σ-algebra of all Borel sets in G. A probability measure μ is a nonnegative countably additive function on $\mathcal{B}$ such that the total mass $\mu(G) = 1$. A point mass measure or Dirac delta measure is a measure μ for which there is an element $x \in G$ such that $\mu(A) = 1$ if $x \in A$ and zero otherwise; $A \in \mathcal{B}$. Such a measure is usually denoted as δ_x. One of the interesting results of Wendel is that

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the only invertible elements in $P(G)$ are Dirac delta measures. The product in $P(G)$ is the convolution $\star$ which is defined as follows;

Definition 2.1. (**Convolution**) Let $\mu, \nu \in P(G)$. Then $\mu \star \nu$ is the probability measure defined as $\mu \star \nu(A) = \int \mu(Ax^{-1})d\nu(x)$ for every $A \in \mathcal{B}$.

Definition 2.2. (**Generalised inverse**) Let $\mathcal{S}$ be a semigroup and let $s \in \mathcal{S}$. An element $s^\dagger \in \mathcal{S}$ is called a generalised inverse of s if $ss^\dagger s = s$.

For example, it is well known that the set $M_N(\mathbb{C})$ of all complex matrices of finite order N is a regular semi-group. The property regularity is almost essential in the fundamental characterization theorems of KSS Nambooripad [14]. In this short note, we do not analyze the implications and consequences of Nambooripad's theory in this concrete semi-group which is postponed to a different project altogether.

3. Regularity Question

To avoid confusion we will adopt the following convention.For a measure μ,the usual regularity in the measuretheritic sense will be addressed as *REGULAR* and the algebraic regularity will be addressed as "regular".

For a compact topological group G, J.G. Wendel [16] proved that **the set $P(G)$ is a semi-group which is not a group under convolution by proving that the only invertible elements in $P(G)$ are Dirac delta measures.** One crucial property needed for measures under consideration is the **regularity** which is not guaranteed for compact topological groups. Next, we quote a basic theorem due to Wendel.

Definition 3.1. (**Support**) Support of $\mu \in P(G)$ is defined as $supp(\mu) = \{g \in G : \mu(E_g) > 0$ for every neighbourhood E_g of $g \in G\}$.

Theorem 3.2 (Wendel)**.** *Let A and B be supports of two measures μ and ν in $P(G)$.Then $supp(\mu \star \nu) = AB = \{xy | x \in A, y \in B\}$*

Now we prove the main theorem of this short research article.

Theorem 3.3. *Let G be a nontrivial compact topological group. Then $P(G)$ is not regular.*

Proof. First we prove the assertion for the special case for which group G is such that $a^2 \neq e$ for some $a \in G$. Let $a \in G$ be such that $a^2 \neq e$.Consider the probability measure $\mu = \frac{\delta_e + \delta_a}{2}$ where δ_x is the Dirac delta measure at x for each $x \in G$.We show that μ does not have a generalised inverse. Let if possible a generalised inverse $\mu^\dagger$ of μ exist. Therefore we have

$$\mu \star \mu^\dagger \star \mu = \mu. \tag{3.1}$$

and $\mu \star \mu^\dagger$ is an idempotent. Clearly $supp(\mu) = \{e, a\}$ and H= $supp(\mu \star \mu^\dagger)$ is a compact subgroup of G by Theorem 1 in [1]. Now combining Wendels's theorem and equation 3.1 we find that

$$H.\{e, a\} = \{e, a\} \tag{3.2}$$

Let $h \in H$ and the equation 3.2 above implies that

$$h.\{e, a\} \subset \{e, a\} \Rightarrow he = e \quad or \quad he = a \text{ and } \quad ha = e \quad or ha = a. \tag{3.3}$$

Now $he = e \Rightarrow h = e$ or $h = a$. Again $ha = e \Rightarrow h = a^{-1}$ or $ha = a \Rightarrow h = e$.

Thus to summarise we get $h = e \quad or \quad h = a^{-1}$. Thus the possibilities are $h = e$ for all $h, \{h = e, h = a^{-1}\}$,$\{h = e, h = a\}$. Thus we get $H = \{e\}$ or $H = \{e, a\}$ or $H = \{e, a^{-1}\}$. Now H is a group. The second and third option would imply that $a^2 = e$ which is against the hypothesis.Therefore we have $H = \{e\}$. Now $\mu \star \mu^\dagger$ is a projection and therefore we get $\mu \star \mu^\dagger = \delta_e$, which is the identity.Thus μ is right invertible Let $supp(\mu^\dagger) = F$. Observe that $supp(\mu \star \mu^\dagger) = \{e\}$. Therefore we have $\{e, a\}.F = \{e\}$. Let $f \in F$.Then $e.f = f = e$ and $a.f = e \Rightarrow a = e$, which is again not possible. All these absurd conclusions are consequence of the assumption that μ is regular.

Now let G be such that $a^2 = e$ for every $a \in G$. Let $a \neq e$. Consider $\mu = \alpha_0\delta_e + \alpha_1\delta_a$,where $0 \leq \alpha_0, \alpha_1 \leq 1, \alpha_0 + \alpha_1 = 1$.Then we have $\mu \in P(G)$ and $supp(\mu) = \{e, a\}$. First we show that μ is an idempotent if and only if $\alpha_0 = \alpha_1 = \frac{1}{2}$.

Observe that $\mu^2 = (\alpha_0^2 + \alpha_1^2)\delta_e + 2\alpha_0\alpha_1\delta_a$. Therefore $\mu^2 = \mu$ if and only if $\alpha_0^2 + \alpha_1^2 = \alpha_0$ and $2\alpha_0\alpha_1 = \alpha_1$, if and only if $\alpha_0 = \alpha_1 = \frac{1}{2}$. Now, let if possible, μ for which $\alpha_0 \neq \frac{1}{2}$ has a generalised inverse $\mu^\dagger$. It is an easy consequence of Wendel's support theorem that $\mu^\dagger = \beta_0\delta_e + \beta_1\delta_a$ where $0 \leq \beta_0, \beta_1 \leq 1, \beta_0 + \beta_1 = 1$; the proof is as follows. Let $H = supp(\mu^\dagger)$. We have by Wendel's theorem

$$\{e, a\}.H.\{e, a\} = \{e, a\} \tag{3.4}$$

Let $h \in H$. Then $h \in \{e, a\}$.Thus $supp(\mu^\dagger) \subseteq \{e, a\}$.

Now we have that $\mu \star \mu^\dagger$ is an idempotent. But an easy computation shows that

$$\mu \star \mu^\dagger = (\alpha_0\beta_0 + \alpha_1\beta_1)\delta_e + (\alpha_0\beta_1 + \alpha_1\beta_0)\delta_a \tag{3.5}$$

Therefore we must have $(\alpha_0\beta_0 + \alpha_1\beta_1) = \frac{1}{2} = (\alpha_0\beta_1 + \alpha_1\beta_0)$. Solving the above linear equations we obtain $\alpha_0 = \frac{1}{2} = \alpha_1$ provided $\beta_0 \neq \beta_1$. Now assume that $\beta_0 = \beta_1$.This means that $\beta_k = \frac{1}{2}$ for all k. Now we use the full force of generalised inverse as follows. We have

$$[(\alpha_0\beta_0 + \alpha_1\beta_1)\delta_e) + (\alpha_0\beta_1 + \alpha_1\beta_0)\delta_a] \times (\alpha_0\delta_e + \alpha_1\delta_a) = \alpha_0\delta_e + \alpha_1\delta_a$$

$$\Rightarrow \frac{\alpha_0 + \alpha_1}{2} = \alpha_0 \Rightarrow \alpha_0 = \frac{1}{2} = \alpha_1$$

Therefore for $\alpha_k \neq \frac{1}{2}, 0 \leq \alpha_0, \alpha_1 \leq 1, \alpha_0 + \alpha_1 = 1$ $\alpha_0\delta_e + \alpha_1\delta_a$ will not be regular.This completes the proof. □ □

Remark 3.4. The above regularity problem was stated and left open in [12]. Wendel proved that the only invertible elements are Dirac delta measures at various points. The problem of characterizing regular elements of $P(G)$ seems interesting. We do not address this problem here. Observe that towards the end of the proof of the above theorem we actually solved this question for a very special case for which $G = \{e, a\}$. In fact we prove that the only regular elements of $P(G)$ are $\{\delta_e, \delta_a, \frac{\delta_e+\delta_a}{2}\}$.

Remark 3.5. The set $P(G)$ is a closed convex set under weak* topology of measures and $\{\delta_g : g \in G\}$ is the set of extreme points of $P(G)$. Hence by Krein-Millmann theorem, the closed convex hull $\overline{conv}\{\delta_g : g \in G\} = P(G)$. In particular, if one consider the subsemi-group $conv\{\delta_g : g \in G\}$, it may be possible to locate all regular elements in it geometrically. This possibility is under investigation.

4. Embedding $P(G)$ in Regular Semigroups

Our next goal is to embed $P(G)$ in larger semigroups in an optimal way. To do this we use non-commutative Fourier transform techniques.

4.1. **Non-Commutative Fourier Transforms.** Recall that for a locally compact topological group G ,$\widehat{G}$ will denote the unitary dual space of G. More explicitly

$$\widehat{G} = \{(\pi, H_\pi)\} \tag{4.1}$$

where $\pi : G \to B(H_\pi), \pi$ is unitary, irreducible representation of G on a complex separable Hilbert space H_π with the identification by unitary equivalence of representations. It is also well-known that when G is compact, then each H_π is finite dimensional. That means the dimension d_π of H_π is finite and $d_\pi = 1$ if G is abelian. The Fourier transform of a $\mu \in P(G)$ is defined as a function on $\hat{\mu} : \widehat{G} \to B(H_\pi)$ defined by

$$\hat{\mu}(\pi)\psi = \int_G \pi(g^{-1})\psi\mu(dg) \tag{4.2}$$

$\pi \in \widehat{G}$. For a compact, Hausdorff group G let $\mathcal{M} = \cup_{d_\pi} M_{d\pi}(\mathbb{C})$.

A map $\Phi : \widehat{G} \to \mathcal{M}(\widehat{G})$ is called **Compatible** if for each $\pi \in \widehat{G}$, $\Phi(\pi) \in M_{d_\pi}(\mathbb{C})$. Here $M_{d_\pi}(\mathbb{C})$ denotes the set of all $d_\pi \times d_\pi$ complex matrices after identifying with $B(H_\pi)$ for each π.

Recall that the set $\tilde{S}(G) = \{\gamma : \Sigma \to \cup_\pi M_{d_\pi}(\mathbb{C})\}$ is a regular semigroup.

[1] The problem under investigation is the regularity of the following semi-groups and finding the maximal regular subsemigroup of $\widehat{P(G)}$.
[2] The regularity of the associated semi group $\tilde{S}(G)$.
[3] The regularity of the semigroup $\Delta(G)$. Observe that these semigroups are related as follows.

$$\widehat{P(G)} \subset \Delta(G) \subset \tilde{S}(G).$$

Theorem 4.1. *Let G be a compact topological group Then $\tilde{S}(G)$ and $\Delta(G)$ are regular semigroups.*

Proof. It is well known that $\tilde{(S)}(G)$ and $\Delta(G)$ are semi-groups. In either case regularity is easy to establish, as shown below. Let $\gamma \in \tilde{S}(G)$ (or $\Delta(G)$). For each $\pi \in \hat{G}$ let $\gamma^\dagger(\pi)$ be the Moore-Penrose inverse of $\gamma(\pi)$. Clearly $\gamma(^\dagger(\pi) \in \mathcal{M}(\hat{G})$. If $\gamma \in \Delta(G)$ so is $\gamma^\dagger$. □

5. Minimal Regular Semigroups Containing $P(G)$

Next we consider the problem whether there are regular semigroups $\widetilde{\widehat{\Delta(G)}}$ such that

$$\widehat{P(G)} \subset \widetilde{\widehat{\Delta(G)}} \subset \Delta(G). \tag{5.1}$$

We restrict our attention to **compact Lie groups** G where new techniques such as **Log-Ng positivity** [1] are available which is defined as follows:

Definition 5.1. A compatible function $\gamma : \hat{G} \to M$ is called **Lo-Ng positive** if

$$\Sigma_{\pi\in\Omega} d_\pi tr(\pi(g)\gamma(\pi)B(\pi)) \geq 0 \tag{5.2}$$

whenever

$$\Sigma_{\pi\in\Omega} d_\pi tr(\pi(g)B(\pi)) \geq 0 \tag{5.3}$$

for all $g \in G$.

Theorem 5.2. *(Theorem 4.3.2, The Lo-Ng Criterion*[1]*) Let $P(G)$ denote the class of regular probability measures on a compact Lie groupG and $\gamma : G \to \boldsymbol{M}(G)$ be a comptible mapping.Then $\gamma = \hat{\mu}$ if and only if γ is Lo-Ng positive namely*

$$h_n(g) = \Sigma_{\pi \in S_n} z_\pi^{(n)} d_\pi tr(\pi(g)\gamma(\pi)) \geq 0 \tag{5.4}$$

for all $g \in G$,where $\#(S_m), \#(S_n) < \infty$ if $m < n$ and $\pi_0 \in S_n$ for all n.

Remark 5.3. The above theorem is a non-commutative analogue of the celebrated Bochkner's theorem: Let G be a locally compact abelian group and $\hat{G}$ be the dual group of characters. Let $F : \hat{G} \to \mathbb{C}$. Then F is the Fourier transform of a measure μ,

$$\hat{\mu}(\chi) = \int_G \chi(\bar{g})\mu(dg) F(x_i - x_j) \geq 0 \text{ if and only if } F(\hat{e}) = 1,\ F \text{ is continuous at } \hat{e}.$$

$$\hat{\mu}(\pi)\psi = \int_G \pi(g^{-1})\psi\mu(dg),\ \pi \in \hat{G}.$$

6. A few more related questions

Let $\Omega(G)$ be the semi-group of all finite products of idempotents in $P(G)$. There are two questions associated with this.

[1] Is $\Omega(G)$ regular?. If so
[2] Is $\Omega(G)$ the maximal regular semigroup contained in $P(G)$?.
[3] What are the regular elements in $P(G)$?

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