Categories and Free Objects

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February, 2022

Introduction

Tom Leinster author of the book
Basic Category Theory (Cambridge 2014)
begins his introduction as follows.

> *Category theory takes a birds eye view of Mathematics.*
> *From high in the sky details become invisible,*
> *but we can spot patterns that were impossible to detect*
> *from ground level.*

Another quote from
Emily Riehl's *Category Theory in Context* (Dover Modern Maths. 2016)
is the following.

> *Category theory provides a cross disciplinary language for Mathematics designed to delineate general phenomena which enables the transfer of ideas from one area of studies to another.*

Free objects such as free groups, free rings, free semigroups etc. and free products, direct products, maximal quotients etc. appear in several contexts.
A unified description of each of them in various algebraic structures can be provided in the frame work of categories.

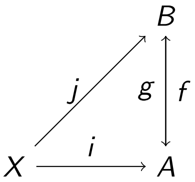
One basic principle in category theory is the use of functions rather than sets in describing mathematical objects.

For example the set A with just one element can be specified as follows.

Theorem 1.1

Let A be a set such that for every set X there is exactly one map from X to A.

We can show that if B is any other set with the above property then B and A are in one to one correspondence.
In the diagram we denote by i and j the unique map in the respective cases.



It follows that

$$fg = 1_A \text{ and } gf = 1_B$$

so that A and B are in one to one correspondence.
Now taking B to be a singleton set we see that A is also singleton.

For another example consider the direct product of two groups G_1 and G_2.

$$G_1 \times G_2 = \{(a, b) : a \in G_1, b \in G_2\}$$

with the natural product.

Also consider the direct product of two topological spaces X and Y.

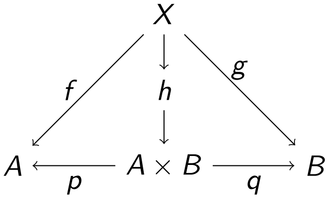
$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

with the product topology.

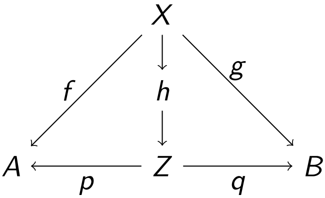
We describe it in terms of related mappings.
That is the projections.
This will show what is common in the above two descriptions.
That is we give a description of the product without explicit description of the product set.

Let us describe the cartesian product of two sets A and B without describing the elements of $A \times B$.
That is to describe it in terms of properties of the projection mappings
$p : A \times B \to A$ and $q : A \times B \to B$.

For any set X and any pair (f, g) of mappings with $f : X \to A$ and $g : X \to B$ there is a unique $h : X \to A \times B$ such that the following diagram is commutative.



We may replace $A \times B$ by any set Z in the above diagram and if



the properties are satisfied.
Then Z and $A \times B$ are in bijective correspondence. So we may consider Z as well as the direct product.

Thus we may define the direct product of A and B as a triple (Z, p, q) satisfying the above conditions.

Advantage This same definition applies to all other contexts as well.

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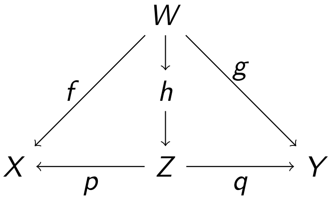
For example the product of two topological spaces X and Y is the space Z with associated continuous maps $p : Z \to X$ and $q : Z \to Y$ such that the conditions as above are satisfied for any space W and contunuous maps f, g.



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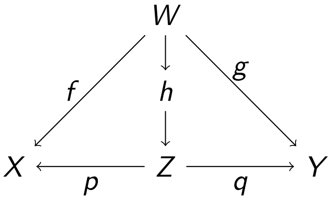


We can see that Z is homeomorphic to the product space $X \times Y$.

Categories

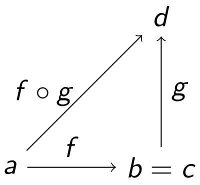
Definition 2.1

A category $\mathcal{C}$ is a pair $(v\mathcal{C}, m\mathcal{C})$ where $v\mathcal{C}$ is a class called the class of objects and $m\mathcal{C}$ is a class called the class of morphisms. The following are the axioms.

- *With each morphism f is associated two objects a, b called the domain and codomain of f respectively.*
  We often write $f : a \to b$.
- *For $a, b \in v\mathcal{C}$ the collection of all morphisms $f : a \to b$ is a set often denoted by $[a, b]_{\mathcal{C}}$. It is also denoted by $m(a, b)$, $Mor(a, b)$, $hom(a, b)$, $[a, b]$ etc.*

Further $[a, b] \cap [c, d] = \emptyset$ unless $a = c$ and $b = d$.

- For $f : a \to b$ and $g : c \to d$ a product $f \circ g$ is defined whenever $b = c$ and $f \circ g : a \to d$.
  Usually $f \circ g$ will denoted as fg.



- For each $a \in v\mathcal{C}$ there is a morphism $1_a : a \to a$ such that

$$1_a f = f \text{ for all } f \text{ with domain } a$$

and

$$g 1_a = g \text{ for all } g \text{ with codomain } a.$$

That is, if $f : a \to b$ then

$$1_a f = f \text{ and } f 1_b = f.$$

- The product is associative. That is

$$f(gh) = (fg)h$$

  whenever the products are defined.

This completes the definition of category.

Remark 2.2

The behaviour of morphisms are similar to that of mappings.
The notation $f : a \to b$ *suggests this.*
A class of sets as objects with mappings between sets as morphisms can be seen as an example of a category.
But in general morphisms in a category need not be mappings.

Example 2.3

A group G may be considered as a category with one object in which all elements of G are the morphisms.
We may denote the object by the identity element e. Then every element $a \in G$ is a morphism

$$a : e \to e.$$

Remark 2.4

A special feature of the above category is that all morphisms are invertible.
Such morphisms in general are called isomorphisms.
That is, a morphism $f : a \to b$ in a category $\mathcal{C}$ is an isomorphism if there is a $g : b \to a$ in $\mathcal{C}$ such that

$$fg = 1_a \text{ and } gf = 1_b.$$

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A category with only one object in which all morphisms are isomorphisms is precisely a group.

Examples

Categories of mathematical structures are considered with the respective structure preserving mappings as the morphisms.
Thus we consider the following examples.
Set – the category of sets with mappins as morphisms
Grp – the category of groups with homomorphisms as morphisms
Ab – the category of abelian groups with homomorphisms
Sgp – the category of semigroups with homomorphisms
Top – the category of topological spaces with continuous maps
$Vect_K$ – the category of vector spaces over a field K with linear maps etc.

Generalizations of Algebraic Concepts

2. Isomorphisms in Categories

Let $\mathcal{C}$ be a category and $f : a \to b$ be a morphism in $\mathcal{C}$. Then f is said to be an isomorphism if there exists $g : b \to a$ such that

$$fg = 1_a \text{ and } gf = 1_b.$$

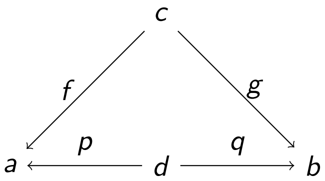
In the category of sets and groups we can see that it is equivalent to f being one to one and onto.
In the category of toplogical spaces an siomorphism is a homeomorphism.
It is more than being a bijection.

2. Direct products in Categories

Let $\mathcal{C}$ be a category and $a, b \in v\mathcal{C}$. A direct product of a and b is a triple (d, p, q) such that

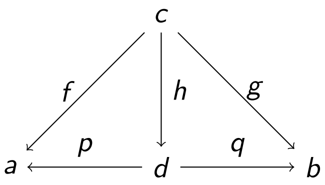
$$a \xleftarrow{\quad p \quad} d \xrightarrow{\quad q \quad} b$$

and if (c, f, g) is any other triple with



then there exists a unique $h : c \to d$ such that

$$f = hp \text{ and } g = hq$$


c
f
h
g
p
q
a
d
b

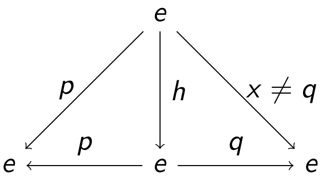
In the category of sets, groups, topological spaces etc. the direct product is the usual one we often see.
As an application we can see that the following theorem in topology on a product space $X \times Y$ is a consequence of the fact that $X \times Y$ is a direct product in the category of topological spaces.

Theorem 4.1

Let X, Y, Z be topological spaces. A map $f : Z \to X \times Y$ is continuous if and only if $f \circ p$ and $f \circ q$ are continuous where p and q are the projections on X and Y respectively.

Remark 4.2

All categories may not admit direct products.
For example let G be a group with more than one element. Then G regarded as a category with one object e does not have a direct product $e \times e$.
A commutative diagram as follows will not exixst.


e
p
h
x ≠ q
p
q
e
e
e

Functors and Natural Transformations

These are often used concepts in category theory.

Structure preserving mappings between categories are called functors.

A functor $F : \mathcal{C} \to \mathcal{D}$ is a pair of mappings both denoted by F such that $F : v\mathcal{C} \to v\mathcal{D}$ and $F : Mor(\mathcal{C}) \to Mor(\mathcal{D})$ satisfying the following.

1. If $f : a \to b$ in $\mathcal{C}$ then $F(f) : F(a) \to F(b)$ in $\mathcal{D}$.

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2. $F(1_a) = 1_{F(a)}$.
3. If $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$ then

$$F(fg) = F(f)F(g).$$

Let $\mathcal{S}et$ denote the category whose objects are sets and morphisms are mappings between them.
For each object a in a category $\mathcal{C}$ the homfunctor $Hom(a, -) : \mathcal{C} \to \mathcal{S}et$ is defined as follows.

For $b \in v\mathcal{C}$ and

$$Hom(a, -)(b) = Hom(a, b) = [a, b]_{\mathcal{C}}.$$

For $b \in v\mathcal{C}$ and

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For convenience we write $H(a, -)$ in place of $Hom(a, -)$.
For $g : b \to c$

$$H(a, g) : H(a, b) \to H(a, c) \text{ maps } f \mapsto fg$$

for all $f \in H(a, b)$.

Natural transformations are morphisms between functors.
Let F, G be functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation $\eta : F \to G$ is a collection $\{\eta_a : a \in v\mathcal{C}\}$ of morphisms in $\mathcal{D}$ such that the following hold.

- For each $a \in v\mathcal{C}, \eta_a$ is from $F(a)$ to $G(a)$.
- For $f : a \to b$ in $\mathcal{C}$

$$\eta_a G(f) = F(f)\eta_b.$$

That is the following diagram is commutative.

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_a} & G(a) \\ F(f)\downarrow & & \downarrow G(f) \\ F(b) & \xrightarrow[\eta_b]{} & G(b) \end{array}$$

For example if $f : b \to a$ in $\mathcal{C}$ then f induces a natural transformation $H(f, -) : H(a, -) \to H(b, -)$ such that for $c \in v\mathcal{C}$

$$H(f, c) : H(a, c) \to H(b, c) \text{ is given by } g \mapsto fg$$

for all $g \in H(a, c)$.

That is the following diagram is commutative.

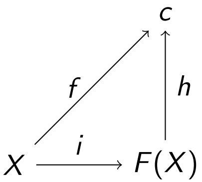
$$\begin{array}{ccc} H(a,c) & \xrightarrow{H(f,c)} & H(b,c) \\ {\scriptstyle H(a,f)}\downarrow & & \downarrow{\scriptstyle H(b,f)} \\ H(a,d) & \xrightarrow[H(f,d)]{} & H(b,d) \end{array}$$

3. Free Objects

The concepts of free groups, free rings etc. are generalised into the concept of free object in a category.
This is applicable only in categories in which objects and morphisms are basically sets and mappings.
That set with some structures as objects and structure preserving mappings as morphisms.

Let $\mathcal{C}$ be such a category and X be a set. A free object on X in $\mathcal{C}$ is a pair $(F(X), i)$ where $F(X) \in v\mathcal{C}$ and $i : X \to F(X)$ is a map such that
and if (c, f) is any other pair with $c \in v\mathcal{C}$ and $f : X \to c$ is a map then there exists a unique morphism $h : F(X) \to c$ such that

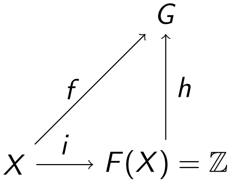
$$f = ih.$$


c
f
h
i
X
F(X)

Free Groups

In the category of groups free object is called free group.
For example if $X = \{a\}$ is a singleton set then the free group on X is the infinite cyclic group.
So is isomorphic to the group $\mathbb{Z}$ of integers.

In the diagram below for any group G and a map $f : X \to G$



$i : X \to \mathbb{Z}$ can be defined by

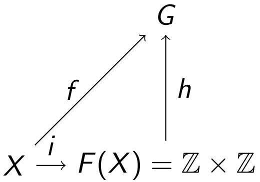
$$i(a) = 1$$

and $h : \mathbb{Z} \to G$ can be defined by

$$h(n) = (f(a))^n.$$

If $X = \{a, b\}$ is a two element set then the free group has a very different structure. In fact it is non abelian.
But if we consider the category Ab of abelian groups, then the free group has a particularly simple structure.

In the category of abelian groups the free group on a set with two elements is isomorphic to the group $\mathbb{Z} \times \mathbb{Z}$.
In the diagram where G is any abelian group



$i : X \to \mathbb{Z} \times \mathbb{Z}$ can be defined by

$$i(a) = (1, 0) \text{ and } i(b) = (0, 1)$$

and $h : \mathbb{Z} \times \mathbb{Z} \to G$ can be defined by

$$h(n, m) = (f(a))^n (f(b))^m.$$

When X is an infinite set the free abelian group on X is **not** the direct product of infinitely many copies of $\mathbb{Z}$.
It is the direct sum of infinitely many copies of $\mathbb{Z}$ indexed by X.

The direct sum

$$\sum_{i \in \mathbb{N}} \mathbb{Z}_i$$

with each $\mathbb{Z}_i = \mathbb{Z}$
has elements
all sequences of integers with all but finitely many terms zero.
Whereas the direct product

$$\Pi_{i \in \mathbb{N}} \mathbb{Z}_i$$

has elements
all sequences of integers.

Theorem 4.3

Free group on any set exists.

We prove it by actual construction of a group which is free on a given set X.

Consider an associated set

$$X^{-1} = \{x^{-1} : x \in X\}$$

where x^{-1} is only a symbol denoting correspondence with x.
We assume

$$X \cap X^{-1} = \emptyset.$$

Now let

$$A = X \cup X^{-1}$$

and let $F(A)$ be the set all words on A. That is

$$w = a_1 a_2 \cdots a_k : a_i \in A.$$

Here each word will look like

$$x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$$

where $x_i \in X$ and λ_i are integers.
Here

$$x^{-2} = (x^{-1})^2.$$

A composition in $F(A)$ is defined by concatenation. That is product of two words w_1 and w_2 is the word

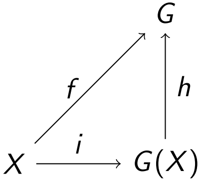
$$w_1 w_2$$

obtained by combining the two words.

Next step is to reduce the given words by removing combinations of the form xx^{-1} and $x^{-1}x$ in each word.
The resulting words are called reduced words.

Now $G(A)$ is taken as the set of all reduced words with composition induced from $F(A)$.
Now $G(A)$ is a group and each element of X appears as a word in $G(A)$.
We can verify that $G(A)$ is the free group on X.

In the diagram where G is any group



$i : X \to G(X)$ can be defined by

$$i(x) = x$$

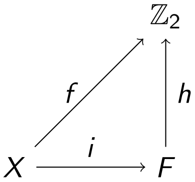
and $h : G(X) \to G$ can be defined by

$$h(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}) = (f(x_1))^{\lambda_1}(f(x_2))^{\lambda_2} \cdots (f(x_k))^{\lambda_k}.$$

This proves that $G(X)$ is a free group.
Thus we get existence of free groups.

It may be noted that free objects may not exist in all categories.
For example consider the category *Fld* of all fields.
Let $X = \{a\}$ be a singleton set.

Let F be any field such that $F \neq \mathbb{Z}_2$ and $i : X \to F$ be any map. Consider the diagram with $f(a) = 1$.



No such h exists as there is no homomorphism from F to $\mathbb{Z}_2$.

Free Semigroups

The free semigroup on a singleton set is isomorphic to the semigroup $\mathbb{N}$ of natural numbers under addition.

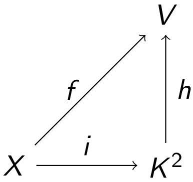
The free monoid on a singleton set is isomorphic to the semigroup $\mathbb{N} \cup \{0\}$ under addition.

The free semigroup on an arbitrary set A is the semigroup of all words on A with concatenation as the binary operation.
If empty word also is included we get the free monoid.

Free Vector spaces

Consider the category $Vect_K$ of all vector spaces over a field K.
Let $X = \{x_1, x_2\}$ be a set of two elements.
Then the free vector space on X over the field K is isomorphic to the space K^2.

Let V be any vector space over K and $f : X \to V$ be any map. Consider the diagram with $i(x_1) = (1, 0)$ and $i(x_2) = (0, 1)$.



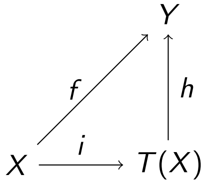
We can consider

$$h(a, b) = af(x_1) + bf(x_2).$$

The free vector space on X can be considered as the vector space with X as a basis.
Note that it contains only finite linear combinations.

Free Topological Spaces

Let X be any set and $T(X)$ be the discrete space on X.
Let Y be any topological space and $f : X \to Y$ be any map.
Consider the diagram with $i(x) = x$.



We can consider

$$h(x) = f(x).$$

Since $T(X)$ is discrete all mappings from $T(X)$ are continuous.

General Questions on Groups, Semigroups etc.

1. Given a group G determine whether it is free.

For example we can prove that no finite group is free.

Is the generalized linear group $GL(n, K)$ free.

Observe that if

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

then $A^2 = I$.
So $GL(2, K)$ is not free.

2. Word Problem

Every group is a homomorphic image of a free group.
Equivalently every group is isomorphic to a quotient of a free group.
This suggests the possibility that every group can be given a presentation with generators and relations.

The **word problem** is the problem of determining when two words on the generating set are equal in the group.
That is determining whether two given words are related.

For example the Klein Four group can be presented as

$$\langle a, b : a^2 = 1,\ b^2 = 1,\ ab = ba \rangle$$

Here the determination of related words is not difficult.

For example consider the following pairs of words.

$$aba^2b^3ab \text{ and } ab^3a^4baba$$

The first word reduces to a^4b^5 and then to b.
The second word reduces to a^7b^5 and then to ab.
Conclusion The words represent different elements.

Now consider

$$baba^2b^3a^4b \text{ and } b^3ab^4a^3ba$$

Here the first word reduces to a^7b^6 and then to a.
The second word reduces to a^5b^8 and then to a.
Conclusion The words represent same elements.

As another example consider the Symmetric group S_3.
It has a presentation

$$\langle a, b : a^2 = 1,\ b^3 = 1,\ ab = b^2 a \rangle$$

Here the determination of related words needs more steps.

It can be observed that the assignment

$$X \mapsto F(X)$$

where $F(X)$ is the free object in a category $\mathcal{C}$ is a functor from the category of sets to $\mathcal{C}$.