Cross Connections and Example

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1 Introduction

The cross connection theory for the structure of regular semigroups introduced by KSS Nambooripad is discussed here with details on an example. The example is a four element band S. The normal category $\mathcal{L}(S)$ of principal left ideals and the normal category $\mathcal{R}(S)$ of principal right ideals are described. It is shown that the semigroup of normal cones $T\mathcal{L}(S)$ and $T\mathcal{R}(S)$ are different and non isomorphic to the band S.

2 Categories

- A category (to be more precise, a small category) is a pair $(v\mathcal{C},\ m\mathcal{C})$ where $v\mathcal{C}$ is called the set of objects and $m\mathcal{C}$ is called the set of morphisms. With each $f \in m\mathcal{C}$ is associated two objects a, called the domain of f, and b, called the codomain of f. We denote this relation by writing $f : a \to b$. We denote the set of all morphisms with domain a and codomain b as

$$m(a, b) \text{ or } [a, b]_{\mathcal{C}} \text{ or } hom(a, b).$$

Further there is a composition $m(a, b) \times m(b, c) \to m(a, c)$ such that

- $f(gh) = (fg)h$, whenever the products are defined.
- Every morphism $f \in m(a, b)$ has a right identity 1_b and a left identity 1_a such that

$$f1_b = f \text{ and } 1_af = f$$

We usually denote a category $(v\mathcal{C}\ , m\mathcal{C})$ by $\mathcal{C}$ only. Also we write $\mathcal{C}$ to denote the morphism class $m\mathcal{C}$ so that $f \in \mathcal{C}$ means that f is a morphism in $\mathcal{C}$.

The structure preserving mappings between categories are called functors. The following is the definition.

Definition 1. *Let $\mathcal{C}, \mathcal{D}$ be categories. A functor F from $\mathcal{C}$ to $\mathcal{D}$ is a pair of mappings, one from $v\mathcal{C}$ to $v\mathcal{D}$ and the other from $m\mathcal{C}$ to $m\mathcal{D}$ (both denoted by F for convenience) satisfying the following.*

- *If $f : a \to b$ in $\mathcal{C}$ then $F(f) : F(a) \to F(b)$ in $\mathcal{D}$.*
- *For each $a \in v\mathcal{C}$*
$$F(1_a) = 1_{F(a)}.$$
- *For $f : a \to b$ and $g : b \to c$ in $\mathcal{C}$*
$$F(fg) = F(f)F(g).$$

Another concept that arises in the discussion on categories is that of natural transformations. These are relations between functors described as follows.

Definition 2. *Let $\mathcal{C}, \mathcal{D}$ be categories and F, G be functors from $\mathcal{C}$ to $\mathcal{D}$. A natural transformation $\eta : F \to G$ is a collection $\{\eta_a : a \in v\mathcal{C}\}$ of morphisms in $\mathcal{D}$ such that the following hold.*

- *For each $a \in v\mathcal{C}, \eta_a$ is from $F(a)$ to $G(a)$.*
- *For $f : a \to b$ in $\mathcal{C}$*
$$\eta_a G(f) = F(f)\eta_b.$$

That is the following diagram is commutative.

$$\begin{array}{ccc} F(a) & \xrightarrow{\eta_a} & G(a) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(b) & \xrightarrow[\eta_b]{} & G(b) \end{array}$$

Example 1. *Examples of functors and natural transformations that occur frequently in discussions on categories are the following. Let $\mathcal{C}$ be a category and Set be the category of sets with usual mappings as morphisms. For each object a in $\mathcal{C}$ a functor $Hom(a,-) : \mathcal{C} \to Set$ is defined as follows. For $c, d \in v\mathcal{C}$ and $f : c \to d$*

$$Hom(a,-)(c) = Hom(a,c) = [a,c]_{\mathcal{C}}$$

and $Hom(a,-)(f) = Hom(a,f) : Hom(a,c) \to Hom(a,d)$ is defined by

$$h \mapsto hf$$

for all $h : a \to c$. These functors are called homfunctors. There is a naturally defined natural transformation between these homfunctors. Let $Hom(a,-)$ and $Hom(b,-)$ be homfunctors and $g : b \to a$ be a morphism. Then $Hom(g,-) : Hom(a,-) \to Hom(b,-)$ is a natural transformation defined by

$$Hom(g,-)_c = Hom(g,c) : Hom(a,c) \to Hom(b,c) \text{ mapping } h \mapsto gh.$$

It is easy to see that every natural transformation from $Hom(a,-)$ to $Hom(b,-)$ is determined by a morphism $g : b \to a$.

Theorem 1. *Let $\eta : Hom(a,-) \to Hom(b,-)$ be a natural transformation. Then there exists a morphism $g : b \to a$ such that $\eta = Hom(g,-)$.*

Definition 3 (Monomorphism, Epimorphism, and Isomorphism)**.** *Let $\mathcal{C}$ be a category and $f \in \mathcal{C}$.*

- *f is said to be a monomorphism if $gf = hf$ implies $g = h$ for any morphisms $g, h \in \mathcal{C}$.*
- *f is said to be an epimorphism if $fg = fh$ implies $g = h$ for any morphisms $g, h \in \mathcal{C}$.*
- *An isomorphism is a morphism which is both left and right invertible. That is $f : a \to b$ is an isomorphism if and only if there is $g : b \to a$ such that*

$$fg = 1_a \text{ and } gf = 1_b.$$

3 Normal Category

KSS Nambooripad introduced normal categories starting with the concept of category with subobjects. This induces a partial order on the object set $v\mathcal{C}$ and inclusion morphisms from a to b for $a \leq b$.

Here we modify the description with a directly assigned partial order on the object set of the category and associating an inclusion morphism from the smaller object to the bigger one.

A category with this partial order together with the factorization is taken as a category with normal factorization.

Definition 4 (Category with normal factorization)**.** *A category with normal factorization is a small category C with the following properties.*

- *The vertex set vC of C is a partially ordered set such that whenever $a \leq b$ in vC, there is a monomorphism $j(a,b) : a \to b$ in C. This morphism is called the inclusion from a to b.*
- *$j : (vC, \leq) \to C$ is a functor from the preorder $(vC, \leq)$ to C which maps (a,b) to $j(a,b)$ for $a, b \in vC$ with $a \leq b$.*
- *For $a, b \leq c$ in vC, if*
$$j(a,c) = fj(b,c)$$
for some $f : a \to b$ then $a \leq b$ and $f = j(a,b)$.
- *Every morphism $j(a,b) : a \to b$ has a right inverse $q : b \to a$ such that*
$$j(a,b)q = 1_a.$$
Such a morphism q is called a retraction in C.
- *Every morphism f in C has a factorization*
$$f = quj$$
where q is a retraction, u is an isomorphism and j is an inclusion. Such a factorization is called normal factorization in C.

Proposition 1 (Epimorphic Component and Image)**.** *Let $\mathcal{C}$ be a category with normal factorization. If*

$$f = quj \text{ and } f = q'u'j'$$

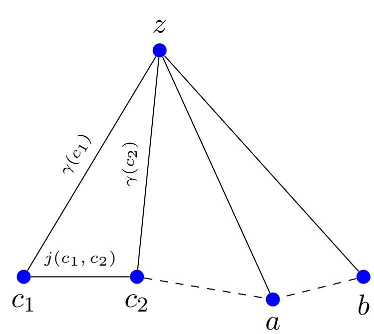
are two normal factorizations of $f \in \mathcal{C}$ then $j = j'$ and $qu = q'u'$.

- In this case $f^\circ = qu$ is called the epimorphic component of f.
- The codomain of f° is called the image of f and is denoted by im f

An important feature of normal categories is the existence of clusters of morphisms called normal cones. A normal cone is defined as follows.

Definition 5 (Normal Cones)**.**
- *A* cone γ *with vertex* z *is a function from* vC *to* mC*, satisfying the following*
  - $\gamma(c) \in C(c, z)$ *for all* $c \in vC$
  - *If* $c_1 \subseteq c_2$ *then* $\gamma(c_1) = j(c_1, c_2)\gamma(c_2)$
- *If there exist* $d \in vC$ *such that* $\gamma(d)$ *is an isomorphism, then* γ *is called a* normal cone*.*
- *The* M *set of a normal cone* γ *is defined as*

$$M\gamma = \{d \in vC : \gamma(d) \text{ is an isomorphism}\}$$



Now we define normal categories.

Definition 6 (Normal Categories)**.** *A normal category is a category* $\mathcal{C}$ *with normal factorization such that for each* $a \in v\mathcal{C}$ *there is a normal cone* γ *with* $\gamma(a) = 1_a$*.*

Here also we deviate slightly from the way in which KSS Nambooripad had given the condition on existence of normal cones in the definition of normal categories. In [4] a normal category is defined as a category with normal factorization in which for every object c there exists a normal cone γ such that $\gamma(c)$ is an isomorphism. Now we show that both these descriptions are equivalent.

Proposition 2. *Let $\mathcal{C}$ be a category with normal factorization. Then the following are equivalent.*

- *For each $a \in v\mathcal{C}$ there is a normal cone γ with $\gamma(a) = 1_a$.*
- *For each $a \in v\mathcal{C}$ there is a normal cone γ such that $\gamma(a)$ is an isomorphism.*

Example 2 (Normal Category: An Example)**.** *One simple example of a normal category is the category $C(X)$ of all proper subsets of a set X.*

Here morphisms are mappings between the sets.

The partial order on objects is the usual inclusion in sets.

The inclusion morphism is the usual inclusion map.

Clearly inclusion map is a monomorphism.

To see that every inclusion $j : a \to b$ has a right inverse consider $a \subseteq b$. Fix an element $z \in a$. Define $q : b \to a$ by

$$q(x) = \begin{cases} x \text{ if } x \in a \\ z \text{ if } x \in b \text{ and } x \notin a. \end{cases}$$

Clealy

$$jq = 1_a.$$

Thus every inclusion has a right inverse. We may verify the remaining axioms.

For $a, b, c \in vC(X)$ let $a, b \leq c$ and $f : a \to b$ be such that

$$j(a, c) = fj(b, c).$$

Now for any $x \in a$

$$(x)j(a, c) = x \text{ and } (x)fj(b, c) = (x)f.$$

So $(x)f = x$ for all $x \in a$.

That is $a \subseteq b$ and f is an inclusion.

Normal factorization is easy to see. For if $f : a \to b$ is a map then Choose b_0 to be the image of f and a_0 to be a cross section of

$$ker f = \{(x, y) : f(x) = f(y)\}.$$

Then $a_0 \leq a$ and $b_0 \leq b$.

Now $q : a \to a_0$ be any extension to a of the identity map on a_0. Let u be the restriction of f to a_0 and $j = j(b_0, b)$. Then

$$f = quj$$

and this is a normal factorization of f.

It may be noted that here a_0 and q have several choices possible and thus the factorization is not unique.

But b_0 is the image of f and so is fixed by f.

Consequently the j in the factorization is also unique.

To conclude that $C(X)$ is a normal category it remains to show that normal cones as required are available.

For any $a \in vC(X)$ consider a map $\alpha : X \to a$ which is onto. Defining $\gamma(b)$ to be the restriction of α to b, that is

$$\gamma(b) = \alpha|b \text{ for all } b \in vC(X)$$

we see that γ is a normal cone in $C(X)$ with vertex a.

Choosing α to be an extension of the identity map on a to a map from X to a we see that the induced normal cone γ has the property that

$$\gamma(a) = 1_a.$$

Thus $C(X)$ is a normal category.

3.1 Normal Category: The General Example

The category of Principal Left Ideals $\mathcal{L}(S)$ of a regular semigroup S is the general example of a normal category. In fact we can see that every normal category arises as the category $\mathcal{L}(S)$ of principal left ideals of a regular semigroup S.

We observe the following.

- The concept of normal category arises as an abstraction of the category of principal left[resp. right] ideals of a regular semigroup with properly defined morphisms.

- Let S be a regular semigroup. The category of principal left ideals $\mathcal{L}(S)$ is defined as follows.

  - $v\mathcal{L}(S) = \{Se : e \in E(S)\}$ where $E(S)$ is the set of idempotents of S. Since S is a regular semigroup every principal left ideal is generated by an idempotent and so $v\mathcal{L}(S)$ is the set of all principal left ideals of S.

- A morphism from Se to Sf in $\mathcal{L}(S)$ is a right translation ρ_u induced by an element $u \in eSf$ and is denoted by $\rho(e,u,f) : Se \to Sf$, defined by $x \mapsto xu$ for all $x \in Se$.

- The identity from Se to Se is $\rho(e,e,e)$
- The compositions are defined by

$$\rho(e,u,f)\rho(f,v,g) = \rho(e,uv,g).$$

The normal category structure on $\mathcal{L}(S)$ is provided by considering the partial order on $v\mathcal{L}(S)$ as usual inclusion and inclusion morphism as usual inclusion mapping.

We observe that if $Se, Sf \in v\mathcal{L}(S)$ and $Se \subseteq Sf$ then $\rho(e,e,f)$ is a morphism in $\mathcal{L}(S)$ and $\rho(e,e,f)$ maps

$$x \mapsto xe = x \text{ for all } x \in Se.$$

Thus the usual inclusion from Se to Sf is a morphism in $\mathcal{L}(S)$.

In this case we can also show that this inclusion from Se to Sf has a right inverse in $\mathcal{L}(S)$. Since $Se \subseteq Sf$ we see that $g = fe \in fSe$ and $Sg = Se$ and so $\rho(f,g,g) = \rho(f,g,e) : Sf \to Se$ in $\mathcal{L}(S)$. Now for every $x \in Se$

$$x(\rho(e,e,f)\rho(f,g,e)) = x(\rho(e,e,f)\rho(f,fe,e)) = xefe = xe = x.$$

Thus $\rho(f,g,g) = \rho(f,g,e)$ is a right inverse of the inclusion $\rho(e,e,f)$.

The general properties of morphisms in $\mathcal{L}(S)$ are listed in the following theorem.

Theorem 2. *Let $\mathcal{L}(S)$ be the normal category given above. Then the following hold.*

- *$\rho(e,u,f) = \rho(e',u',f')$ if and only if $e\mathcal{L}e'$, $f\mathcal{L}f'$ and $u' = e'u$.*
- *$\rho(e,u,f)$ is a monomorphism if and only if $\rho(e,u,f)$ is injective and this is true if and only if $e\mathcal{R}u$.*
- *$\rho(e,u,f)$ is an epimorphism if and only if $\rho(e,u,f)$ is surjective and this is true if and only if $u\mathcal{L}f$.*
- *Se and Sf are isomorphic if and only if $e\mathcal{D}f$ and $\rho(e,u,f)$ is an isomorphism if and only if $e\mathcal{R}u\mathcal{L}f$.*

- *If $Se \subseteq Sf$ then $j(Se, Sf) = \rho(e, e, f)$ and $\rho(f, v, e)$ is a retraction if and only if $\rho(f, v, e) = \rho(f, g, g)$ for some idempotent g such that $g \leq f$ and $Sg = Se$.*

Now we show that every morphism in $\mathcal{L}(S)$ admits a normal facorization.

Theorem 3. *Every morphism in $\mathcal{L}(S)$ has a normal factorization and every normal factorization of $\rho(e, u, f)$ is of the form*

$$\rho(e, u, f) = \rho(e, g, g)\rho(g, u, h)\rho(h, h, f) \tag{1}$$

where $h \in E(L_u)$ and $g \in E(R_u) \bigcap \omega(e)$. Here $\rho(e, g, g)$ is a retraction, $\rho(g, u, h)$ is an isomorphism and $\rho(h, h, f)$ is an inclusion.

When S is a regular semigroup $E(R_u) \bigcap \omega(e) \neq \emptyset$ for all $e \in E(S)$ and $u \in eSf$ with $f \in E(S)$. Therefore $\mathcal{L}(S)$ is a category with normal factorization.

3.2 Normal Cones in $\mathcal{L}(S)$

Let S be a regular semigroup and $\mathcal{L}(S)$ be the category of principal left ideals of S. We show that several normal cones exist in $\mathcal{L}(S)$.

For each $a \in S$ we describe a normal cone ρ^a in $\mathcal{L}(S)$ with vertex $Sf = Sa$ as follows. For each $Se \in v\mathcal{L}(S)$

$$\rho^a(Se) = \rho(e, ea, f).$$

Now if $Sg \subseteq Se$ then

$$\rho^a(Sg) = \rho(g, ga, f) = \rho(g, g, e)\rho(e, ea, f) \text{ since } gea = ga \text{ as } ge = g.$$

That is $\rho^a(Sg) = j(Sg, Se)\rho^a(Se)$. Choosing an idempotent h such that $h\mathcal{R}a$ we see that

$$h\mathcal{R}a\mathcal{L}f$$

and so $\rho(h, a, f)$ is an isomorphism. Since $h\mathcal{R}a$ we have $ha = a$ and so

$$\rho^a(Sh) = \rho(h, ha, f) = \rho(h, a, f)$$

is an isomorphism. Thus ρ^a is a normal cone for each $a \in S$.

Choosing $a = f$ we see that the normal cone ρ^f has the propery that

$$\rho^f(Sf) = \rho(f, f, f)$$

is the identity at Sf. This completes the verification that $\mathcal{L}(S)$ is a normal category.

3.3 All normal categories are $\mathcal{L}(S)$

Now we proceed to show that every normal category arises as $\mathcal{L}(S)$ of a regular semigroup S. First we build up a regular semigroup from a normal category. This is the semigroup of all normal cones in the category. To be precise consider a normal category $\mathcal{C}$. Let $T\mathcal{C}$ denote the set of all normal cones in $\mathcal{C}$. Define a product in $T\mathcal{C}$ as follows. For $\gamma, \delta \in T\mathcal{C}$ the product $\gamma\delta$ is the normal cone whose components are given by

$$(\gamma\delta)(a) = \gamma(a)(\delta(c_\gamma))^o$$

where c_γ is the vertex of γ and the notation ^{o} denotes the epimorphic component. It is easy to see that $\gamma\delta$ is a normal cone and that the product is associative.

The following result is often found useful.

Lemma 1. *Let γ be a normal cone in a normal category $\mathcal{C}$. For any epimorphism $g : c_\gamma \to c$ there is a normal cone $\gamma * g$ whose components are*

$$(\gamma * g)(a) = \gamma(a)g.$$

Proof. Now we show that $\gamma * g$ is a normal cone.

First we show that for $b \leq a$ in vC

$$(\gamma * g)(b) = j(b,a)(\gamma * g)(a).$$

Now

$$\begin{aligned} j(b,a)(\gamma * g)(a) &= j(b,a)(\gamma(a)g) \\ &= \gamma(b)g = (\gamma * g)(b). \end{aligned}$$

Next we show that there is an object d such that

$$(\gamma * g)(d) \text{ is an isomorphism.}$$

Since γ is a normal cone there is $b \in vC$ such that $\gamma(b)$ is an isomorphism. Now $\gamma(b)g$ is an epimorphism and so has a normal factorization as

$$\gamma(b)g = qu$$

for a retraction $q : b \to b_0$ and an isomorphism $u : b_0 \to c$ for some $b_0 \leq b$. So

$$\begin{aligned} (\gamma * g)(b_0) = \gamma(b_0)g &= j(b_0,b)\gamma(b)g \\ &= j(b_0,b)qu = u \end{aligned}$$

since $q : b \to b_0$ is a retraction. Thus there is a component $(\gamma * g)(b_0)$ which is an isomorphism. So $\gamma * g$ is a normal cone. □

Using this lemma we see that $T\mathcal{C}$ is a semigroup.

Now we show that $T\mathcal{C}$ is a regular semigroup.

Let $\gamma \in TC$ and let $c_\gamma = c$.
Choose d such that $\gamma(d) = u$ is an isomorphism.
Choose a normal cone σ such that $\sigma(c) = 1_c$.
Choose $\delta = \sigma * u^{-1}$. Then $\delta(c) = u^{-1}$.
Then for any $a \in vC$

$$(\gamma\delta\gamma)(a) = \gamma(a)(\delta(c)\gamma(d))^o$$

$$= \gamma(a)(u^{-1}u)^o = \gamma(a).$$

That is $\gamma\delta\gamma = \gamma$ and so TC is regular.

We can now show that if $\mathcal{C}$ is a normal category then taking $S = T\mathcal{C}$ the normal category $\mathcal{L}(S)$ is isomorphic to $\mathcal{C}$.

The following proposition provides the details of this isomorphism.

Proposition 3. *Let $\mathcal{C}$ be a normal category and $T\mathcal{C}$ be the semigroup of normal cones in $\mathcal{C}$. Then the following hold. For $\gamma, \delta \in T\mathcal{C}$*

- *$\gamma\mathcal{L}\delta$ in $T\mathcal{C}$ if and only if $c_\gamma = c_\delta$.*
- *So the map $S\gamma \mapsto c_\gamma$ is a bijection from $v\mathcal{L}(S)$ to $v\mathcal{C}$.*
- *$S\gamma \subseteq S\delta$ if and only if $c_\gamma \leq c_\delta$.*
- *For idempotents γ and δ in $T\mathcal{C}$ if $\rho(\gamma, \sigma, \delta) : S\gamma \to S\delta$ is a morphism in $\mathcal{L}(S)$ then $\sigma \in \gamma S\delta$ and so $\sigma(c_\gamma) : c_\gamma \to c \leq c_\delta$ for some $c \in v\mathcal{C}$.*
- *For each $g : c_\gamma \to c_\delta,\ \gamma * g^o \in \gamma S\sigma$ and the map*

  $$g \mapsto \rho(\gamma, \gamma * g^o, \delta)$$

  is a bijection from $[c_\gamma, c_\delta]_{\mathcal{C}} \to [S\gamma, S\delta]_{\mathcal{L}(S)}$.
- *The above assignment gives a functor from $\mathcal{C}$ to $\mathcal{L}(S)$.*

Theorem 4. *The categories $\mathcal{C}$ and $\mathcal{L}(S)$ where $S = T\mathcal{C}$ are isomorphic as normal categories.*

3.4 $T\mathcal{L}(S)$ is not S in general

Now we give an example to show that $T\mathcal{L}(S)$ may not be isomorphic to S.

Consider the band $\{a, b, c, d\}$ which is a union of two right zero semigroups $\{a, b\}$ and $\{c, d\}$ with $c \leq a$ and $d \leq b$.As a semilattice of rectangular bands this will appear as follows.

a	b

c	d

It follows that

$$ad = d,\; da = c,\; bc = c \text{ and } cb = d.$$

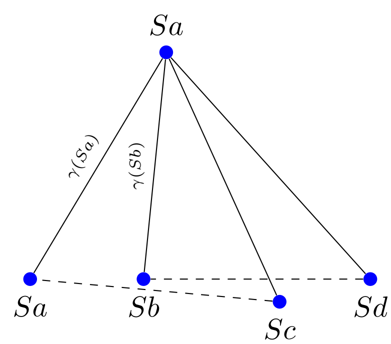
So

$$v\mathcal{L}(S) = \{Sa = \{a, c\},\; Sb = \{b, d\},\; Sc = \{\}, Sd = \{d\}\}$$

and

$$Sc \leq Sa \text{ and } Sd \leq Sb.$$

Now consider all normal cones with vertex Sa.



It can be seen that the cone γ is fully determined by $\gamma(Sa)$ and $\gamma(Sb)$. Since $aSa = \{a, c\}$ there are only two morphism from Sa to Sa. They are

$$\rho(a, a, a) \text{ and } \rho(a, c, a).$$

Similarly since $bSa = \{a, c\}$ there are only two morphism from Sb to Sa and they are

$$\rho(b, a, a) \text{ and } \rho(b, c, a).$$

Since the vertex of γ is at Sa choosing $\gamma(Sa) = \rho(a, c, a)$ and $\gamma(b) = \rho(b, c, a)$ is not possible as there can not exist an isomorphism component in this case. Thus there are three normal cones with vertex Sa. These are

$$\gamma^a = \rho^a, \ \gamma^a_{ac}, \ \text{and } \gamma^a_{bc}$$

where

$$\gamma^a(Sa) = \rho^a(Sa) = \rho(a, a, a) \text{ and } \gamma^a(Sb) = \rho(b, a, a)$$

and

$$\gamma^a_{ac}(Sa) = \rho(a, c, a) \text{ and } \gamma^a_{ac}(Sb) = \rho(b, a, a)$$

and

$$\gamma^a_{bc}(Sa) = \rho(a, a, a) \text{ and } \gamma^a_{bc}(Sb) = \rho(b, c, a).$$

It may be observed that γ^a_{ac} is not an idempotent.

Similarly there are three normal cones with vertex Sb. Let us denote them by

$$\gamma^b = \rho^b, \ \gamma^b_{bd} \text{ and } \gamma^b_{ad}$$

where

$$\gamma^b(Sa) = \rho^b(Sa) = \rho(a, ab, b) = \rho(a, b, b) \text{ and } \gamma^b(Sb) = \rho(b, b, b)$$

and

$$\gamma^b_{bd}(Sa) = \rho(a, b, b) \text{ and } \gamma^b_{bd}(Sb) = \rho(b, d, b)$$

and

$$\gamma^b_{ad}(Sa) = \rho(a, d, b) \text{ and } \gamma^b_{ad}(Sb) = \rho(b, b, b).$$

We observe that γ^b_{bd} is not an idempotent.

Other normal cones in $T\mathcal{L}(S)$ are those with vertex Sc and Sd. Since $aSc = \{c\} = bSc = cSc = dSc$ we see there is only one normal cone with vertex Sc. Similarly there is only one normal cone with vertex Sd. These are ρ^c and ρ^d respectively.

The $\mathcal{D}$-structure of $T\mathcal{L}(S)$ in this case is the following.

$\gamma^a = \rho^a$	$\gamma^b = \rho^b$
γ^a_{ac} *	γ^b_{ad}
γ^a_{bc}	γ^b_{bd} *

ρ^c	ρ^d

Here $*$ indicates that corresponding entry is not an idempotent. Clearly $T\mathcal{L}(S)$ is not isomorphic to S.

3.5 The Semigroup $\mathbf{T}\mathcal{R}(S)$

In this case the normal category $\mathcal{R}(S)$ of principal right ideals and the corresponding $T\mathcal{R}(S)$ are quite different. Here

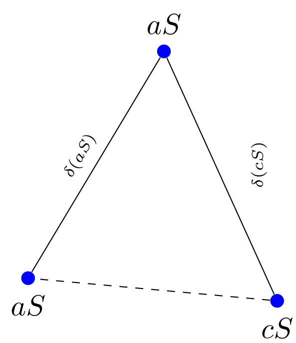
$$v\mathcal{R}(S) = \{aS,\ cS\}$$

where

$$aS = bS = S \text{ and } cS = dS = \{c, d\}.$$

Thus $\mathcal{R}(S)$ has only two objects aS and cS with $cS \leq aS$. Also since $aSa = \{a, c\}$ there are only two morphisms in $\mathcal{R}(S)$ from aS to aS which are $\lambda(a, a, a)$ and $\lambda(a, c, a)$.

Now a normal cone δ with vertex aS is as follows.



Since $cS \leq aS$ the cone is completely determined by $\delta(aS)$. It follows that there is only one normal cone with vertex aS and similarly there is only one normal cone with vertex cS. These cones can be considered as the principal cones λ^a and λ^c. Thus $T\mathcal{R}(S)$ is a two element semilattice $\{\lambda^a,\ \lambda^c\}$ with $\lambda^c \leq \lambda^a$. Note that here $\lambda^b = \lambda^a$ and $\lambda^d = \lambda^c$. Thus $T\mathcal{R}(S)$ is also not isomorphic to S.

4 Cross connections

A cross connection is a relation connecting two normal categories so that one is isomorphic to the normal category $\mathcal{L}(S)$ and the other is isomorphic to the normal category $\mathcal{R}(S)$ of a regular semigroup S. That is a cross connection determines a regular semigroup S with the above properties. The cross connection relation is provided in terms of two functors each from one category into a special dual called normal dual of the other. The usual dual of a category $\mathcal{C}$ is the category of all homfunctors from $\mathcal{C}$ to the category $\mathcal{S}et$ of sets. For normal dual we consider functors which are closely related to homfunctors described in terms of normal cones. These functors are called H-functors.

We begin with the definition of H-functors.

Definition 7. *Let $\mathcal{C}$ be a normal category and γ be a normal cone in $\mathcal{C}$. Then the H-functor $H(\gamma,-) : \mathcal{C} \to \mathcal{S}et$ is defined as follows.*

For $a, b \in v\mathcal{C}$ and $g : a \to b$ in $\mathcal{C}$

$$H(\gamma,-)(a) = H(\gamma, a) = \{\gamma * f^o : f : c_\gamma \to a\}$$

and

$$H(\gamma, g) : H(\gamma, a) \to H(\gamma, b) \text{ is defined by}$$
$$\gamma * f^o \mapsto \gamma * (fg)^o.$$

We can see that each H-functor $H(\gamma,-)$ is naturally isomorphic to the homfunctor $Hom(c,-)$ where $c = c_\gamma$ is the vertex of γ. In fact

$$\eta : H(\gamma,-) \to Hom(c,-)$$

with

$$\eta_a : H(\gamma, a) \to Hom(c, a) \text{ given by } \gamma * f^0 \mapsto f$$

is a natural isomorphism.

Definition 8. *Let $\mathcal{C}$ be a normal category. Then the normal dual of $\mathcal{C}$ is the category $N^*\mathcal{C}$ of all H-functors with natural transformations as morphisms.*

The next theorem gives that $N^*\mathcal{C}$ is a normal category whenever $\mathcal{C}$ is a normal caegory.

Theorem 5. *Let $\mathcal{C}$ be a normal category. Then $N^*\mathcal{C}$ is also a normal category with partial order given by*

$$H(\gamma,-) \leq H(\delta,-) \text{ if } H(\gamma, a) \subseteq H(\delta, a) \text{ for all } a \in v\mathcal{C}.$$

Further the H-functors have the property that $H(\gamma,-) = H(\delta,-)$ if and only if $\gamma\mathcal{R}\delta$. Since $T\mathcal{C}$ is a regular semigroup every $\mathcal{R}$-class contains an idempotent and so every $H(\gamma,-)$ is equal to an $H(\epsilon,-)$ for an idempotent normal cone ϵ. Thus every object of $N^*\mathcal{C}$ is $H(\epsilon, -)$ for an idempotent normal cone ϵ.

It follows that if $H(\gamma, -) = H(\epsilon, -)$ then $M\gamma = M\epsilon$. So we often denote $M\gamma$ by $MH(\gamma, -)$.

Another concept we use in defining cross connection is that of local isomorphism.

For normal categories $\mathcal{C}$ and $\mathcal{D}$, a functor $\Gamma : \mathcal{C} \to \mathcal{D}$ is said to be a local isomorphism if it is full, faithful, order preserving on objects and is an isomorphism on ideals $\langle c \rangle$ for each $c \in v\mathcal{C}$. Here the ideal $\langle c \rangle$ of $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ with object set $\{a \in v\mathcal{C} : a \leq c\}$.

Now we give the definition of cross connection between normal categories.

Definition 9. *[4] Let $\mathcal{C}$ and $\mathcal{D}$ be normal categories. A cross connection is a 4-tuple $(\mathcal{C}, \mathcal{D}, \Gamma, \Delta)$ where $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ and $\Delta : \mathcal{C} \to N^*\mathcal{D}$ are local isomorphisms satisfying the condition*

$$c \in M\Gamma(d) \Longleftrightarrow d \in M\Delta(c)$$

for all $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$.

We often denote a cross connection $(\mathcal{C}, \mathcal{D}, \Gamma, \Delta)$ by (Γ, Δ), if the categories involved are clear in the context.

The cross connection semigroup $S(\Gamma, \Delta)$ associated with a cross connection (Γ, Δ) is described as follows. The functors $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ and $\Delta : \mathcal{C} \to N^*\mathcal{D}$ are realized as functors from the product category $\mathcal{C} \times \mathcal{D}$ to $\mathcal{S}et$ by defining

$$\Gamma(c, d) = (\Gamma(d))(c) \text{ and } \Delta(c, d) = (\Delta(c))(d).$$

The local isomorphism property of Γ and Δ induces a natural isomorphism $\chi : \Gamma \to \Delta$ where Γ and Δ are functors from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{S}et$. The cross connection semigroup is the following.

$$S(\Gamma, \Delta) = \{(\gamma, \delta) : \gamma \in \Gamma(c, d) \text{ and } \delta = \chi_{(c,d)}(\gamma) \in \Delta(c, d)\}$$

for some $(c, d) \in v\mathcal{C} \times v\mathcal{D}$.

The semigroup structure of $S(\Gamma, \Delta)$ arises as a subsemigroup of $T\mathcal{C} \times T^o\mathcal{D}$ where $T^o\mathcal{D}$ is the semigroup on the set $T\mathcal{D}$ of normal cones in $\mathcal{D}$ with the dual of the usual composition as product. Moreover when $(\gamma, \delta) \in S(\Gamma, \Delta)$ we say that (γ, δ) is a linked pair.

Remark 1. *It is possible that a given γ belongs to $\Gamma(c, d)$ and to $\Gamma(c', d')$ for $c \neq c'$ or $d \neq d'$. Then we may get (γ, δ) and (γ, δ') as linked pairs corresponding to the same γ.*

In the next proposition we give some explicit choices of linked pairs of normal cones and a description of the idempotents in $S(\Gamma, \Delta)$.

Proposition 4. *[4] Let (Γ, Δ) be a cross connection between two normal categories $\mathcal{C}$ and $\mathcal{D}$. Then the following hold. Here $c \in v\mathcal{C}$ and $d \in v\mathcal{D}$.*

(i) For each $c \in M\Gamma(d)$ there is a unique idempotent normal cone $\gamma(c, d)$ in $\mathcal{C}$ such that vertex of $\gamma(c, d)$ is c and $\Gamma(d) = H(\gamma(c, d), -)$.

(ii) For each $c \in M\Gamma(d)$ there is a unique idempotent normal cone $\delta(c, d)$ in $\mathcal{D}$ such that vertex of $\delta(c, d)$ is d and $\Delta(c) = H(\delta(c, d), -)$.

(iii) For each $c \in M\Gamma(d)$ the pair $(\gamma(c, d), \delta(c, d))$ is a linked pair and the set of idempotents in the cross connection semigroup $S(\Gamma, \Delta)$ is

$$E(S(\Gamma, \Delta)) = \{(\gamma(c, d), \delta(c, d)) : c \in M\Gamma(d)\}.$$

Example 3. *Consider the band $\{a, b, c, d\}$ given in the previous section. This has two $\mathcal{D}$-classes as follows.*

a	b

c	d

We have seen that

$$ad = d, \; da = c, \; bc = c \text{ and } cb = d.$$

So

$$v\mathcal{L}(S) = \{Sa = \{a, c\}, \; Sb = \{b, d\}, \; Sc = \{c\}, Sd = \{d\}\}$$

and

$$Sc \leq Sa \text{ and } Sd \leq Sb.$$

We recall that the $\mathcal{D}$-structure of $T\mathcal{L}(S)$ in this case is the following.

$\gamma^a = \rho^a$	$\gamma^b = \rho^b$
γ^a_{ac} *	γ^b_{ad}
γ^a_{bc}	γ^b_{bd} *

ρ^c	ρ^d

Here $*$ indicates that corresponding entry is not an idempotent.

We recall that in this case the normal category $\mathcal{R}(S)$ of principal right ideals has only two objects as given below.

$$v\mathcal{R}(S) = \{aS, cS\}$$

and $cS \leq aS$ and $T\mathcal{R}(S)$ is a two element semilattice $\{\lambda^a,\ \lambda^c\}$ with $\lambda^c \leq \lambda^a$.

Now we consider cross connections between the categories $\mathcal{C} = \mathcal{L}(S)$ and $\mathcal{D} = \mathcal{R}(S)$. The natural cross connection (Γ, Δ) is the following.

$$\begin{aligned}
\Gamma(aS) &= H(\epsilon, -) \text{ where } \epsilon = \rho^a \\
\Gamma(cS) &= H(\sigma, -) \text{ where } \sigma = \rho^c \text{ and} \\
\Delta(Sa) &= H(\delta, -) \text{ where } \delta = \lambda^a \\
\Delta(Sb) &= H(\delta, -) \text{ where } \delta = \lambda^a = \lambda^b \\
\Delta(Sc) &= H(\tau, -) \text{ where } \tau = \lambda^c \text{ and} \\
\Delta(Sd) &= H(\tau, -) \text{ where } \tau = \lambda^c = \lambda^d.
\end{aligned}$$

Now

$$\Gamma(Sa, aS) = H(\epsilon, Sa) = \{\epsilon * f^o \text{ where } f : Sa \to Sa\} = \{\rho^a, \rho^c\}$$

$$\Gamma(Sb, aS) = H(\epsilon, Sb) = \{\epsilon * f^o \text{ where } f : Sa \to Sb\} = \{\rho^b, \rho^d\}$$

$$\Gamma(Sc, aS) = H(\epsilon, Sc) = \{\epsilon * f^o \text{ where } f : Sa \to Sc\} = \{\rho^c\}$$

$$\Gamma(Sd, aS) = H(\epsilon, Sd) = \{\epsilon * f^o \text{ where } f : Sa \to Sd\} = \{\rho^d\}$$

Similarly

$$\Gamma(Sa, cS) = H(\sigma, Sa) = \{\sigma * f^o \text{ where } f : Sc \to Sa\} = \{\rho^c\}$$

$$\Gamma(Sb, cS) = H(\epsilon, Sb) = \{\epsilon * f^o \text{ where } f : Sc \to Sb\} = \{\rho^d\}$$

$$\Gamma(Sc, cS) = H(\epsilon, Sc) = \{\epsilon * f^o \text{ where } f : Sc \to Sc\} = \{\rho^c\}$$

$$\Gamma(Sd, cS) = H(\epsilon, Sd) = \{\epsilon * f^o \text{ where } f : Sc \to Sd\} = \{\rho^d\}$$

Also

$$\begin{aligned}
\Delta(Sa, aS) &= H(\delta, aS) = \{\delta * g^o \text{ where } g : aS \to aS\} = \{\lambda^a, \lambda^c\} \\
\Delta(Sb, aS) &= H(\delta, aS) = \{\lambda^a, \lambda^c\} \\
\Delta(Sc, aS) &= H(\tau, aS) = \{\tau * g^o \text{ where } g : cS \to aS\} = \{\lambda^c\} \\
\Delta(Sd, aS) &= H(\tau, aS) = \{\lambda^c\}
\end{aligned}$$

Similarly

$$\begin{aligned}
\Delta(Sa, cS) &= H(\delta, cS) = \{\delta * g^o \text{ where } g : aS \to cS\} = \{\lambda^c\} \\
\Delta(Sb, cS) &= H(\delta, cS) = \{\lambda^c\} \\
\Delta(Sc, cS) &= H(\tau, cS) = \{\tau * g^o \text{ where } g : cS \to cS\} = \{\lambda^c\} \\
\Delta(Sd, cS) &= H(\tau, cS) = \{\lambda^c\}.
\end{aligned}$$

Note that the natural isomorphism $\chi : \Gamma \to \Delta$ has the following components.

$$\begin{array}{cc}
\chi(Sa, aS) : \Gamma(Sa, aS) \to \Delta(Sa, aS) & \text{mapping } \rho^a \mapsto \lambda^a \text{ and } \rho^c \mapsto \lambda^c. \\
\chi(Sa, cS) : \Gamma(Sa, cS) \to \Delta(Sa, cS) & \text{mapping } \rho^c \mapsto \lambda^c \\
\chi(Sb, aS) : \Gamma(Sb, aS) \to \Delta(Sb, aS) & \text{maps } \rho^b \mapsto \lambda^a \text{ and } \rho^d \mapsto \lambda^c. \\
\chi(Sb, cS) : \Gamma(Sb, cS) \to \Delta(Sb, cS) & \text{maps } \rho^d \mapsto \lambda^c.
\end{array}$$

Similary other components can be described. It may be noted that here

$$\lambda^a = \lambda^b \text{ and } \lambda^c = \lambda^d$$

so that the linked pairs and the semigroup of the cross connection can be described as follows.

$$S(\Gamma, \Delta) = \{(\rho^a, \lambda^a), (\rho^b, \lambda^a), (\rho^c, \lambda^c), (\rho^d, \lambda^c)\}.$$

Clearly $S(\Gamma, \Delta)$ is isomorphic to S.

Now we show that in this case there is no other cross connection (Γ_1, Δ_1) between $\mathcal{C} = \mathcal{L}(S)$ and $\mathcal{D} = \mathcal{R}(S)$.

Another choice of a local isomorphism from $\mathcal{D}$ to $N^*\mathcal{C}$ is the following.

$$\begin{aligned}
\Gamma_1(aS) &= H(\gamma, -) \text{ where } \gamma = \gamma^b_{ad} \text{ and} \\
\Gamma_1(cS) &= H(\rho^c, -).
\end{aligned}$$

Now $[aS, aS]_{\mathcal{R}(S)} = \{1_{aS}\}$ and the only natural transformation from $H(\gamma, -)$ to $H(\gamma, -)$ is the identity naural transformation. Also $cS \leq aS$ and $H(\rho^c, -) \leq H(\gamma, -)$. Thus it follows that Γ_1 is a local isomorphism from $\mathcal{R}(S)$ to $N^*\mathcal{L}(S)$.

Yet another choice of a local isomorphism is Γ_2 given by

$$\begin{aligned}
\Gamma_2(aS) &= H(\gamma', -) \text{ where } \gamma' = \gamma^a_{bc} \text{ and} \\
\Gamma_2(cS) &= H(\rho^c, -).
\end{aligned}$$

Now

$$M\Gamma_1(aS) = \{Sb\} \text{ and } M\Gamma_1(cS) = \{Sc\}.$$

Also

$$M\Gamma_2(aS) = \{Sa\} \text{ and } M\Gamma_2(cS) = \{Sc\}.$$

So Sa is not in $M\Gamma_1(d)$ for any $d \in v\mathcal{R}(S)$ and Sb is not in $M\Gamma_2(d)$ for any $d \in v\mathcal{R}(S)$. In order that Γ_1 or Γ_2 give rise to a cross connection we need that both Sa and Sb are in the M-sets of Γ_1 or Γ_2. Thus Γ_1 and Γ_2 fails to be part of a cross connection. Thus we see that there is only one cross connection in this case.

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