IMRT WORKSHOP ON FOUNDATIONS OF COMPLEX ANALYSIS

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Complex Numbers

Qn. What is a non zero number which is not positive and not negative.

Thinking the unthinkable.

ECG Sudarshan

Tachions have no positive or negative mass.

So it has an imaginery mass.

Consider the following question

1. Divide 10 into two parts so that their product is 40.

Geometrically

this means constucting a rectangle whose perimeter is 20 and area is 40.

But we know that the Maximum possible area of a rectangle of perimeter 20 units is 25 sq.units.

If sides are $5 + a$ and $5 - a$ then area is

$$25 - a^2.$$

Quadratic Equations give a solution as follows. Solve

$$x(10 - x) = 40.$$

That is to solve

$$x^2 - 10x + 40 = 0.$$

This gives soluions as

$$5 + \sqrt{-15} \text{ and } 5 - \sqrt{-15}.$$

This was a solution given by Cardano in 1545 Ars Magna.

The first appearence of square root of a negative number in a solution of a real number problem.

Historically complex numbers have originated in the process of solving cubic equations.
Tartaglia - Fiore - Cardano formula for solving cubic of the form

$$x^3 = ax + b$$

where a, b are positive reals is

$$\sqrt[3]{\frac{b}{2} + \sqrt{(\frac{b}{2})^2 - (\frac{a}{3})^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{(\frac{b}{2})^2 - (\frac{a}{3})^3}}$$

Cardano applied this formula to solve

$$x^3 = 15x + 4.$$

Solution is

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Bombelli's Bombshell

$$\sqrt[3]{2+\sqrt{-121}} = 2+\sqrt{-1} \text{ and}$$

$$\sqrt[3]{2-\sqrt{-121}} = 2-\sqrt{-1}.$$

Giving solution

$$x = 4.$$

Story of a Lost Treasure

Gamov in his book *One, Two, Three,... Infinity* narrates the story of a treasure hunt which the hunter lost due his ignorance in complex numbers.

A young man found from his grandfathers's notes the following.

Travel towards north and you will find a deserted island. There is an oak tree and a pine tree on the north shore. Near to it is a hanger tree also. Walk from the hanger tree to the oak tree counting steps. Then turn right 90^o *and walk the same number of steps. Erect a pole at the end. Come back to the hanger tree and walk towards the pine tree counting steps. Then turn* 90^o *towards left and walk the same number of steps. Erect a pole at the end. At the middle of the two poles dig to find the Treasure.*

The young man reached the island and found the oak tree and the pine tree there. But there was no sign of the hanger tree. It was some how distroyed.
He returned feeling unfortunate.
Gamov advices him to learn complex numbers to reach to the treasure.

Solution

This follows from geometry (of complex numbers).

Complex Numbers

There are various ways of representing complex numbers.

- Numbers of the form $a + bi$ where a, b are reals.
- Matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where a, b are reals.

A matrix

$$Z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

may be viewed as

$$Z = aI + bJ$$

where I is the ideniy matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Note that

$$J^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

- Algebraic closure of $\mathbb{R}$ as a field.
- Polynomials in $\mathbb{R}[x]/\langle x^2 + 1\rangle$.

- Ordered pairs of real numbers.
- Points in a plane.
- Vectors represented by length and angle

Geometry of complex plane

The geometric representation of complex numbers as points in the plane was first given by Argand, an accountant and book keeper in Paris in 1806. In his article *Essay on geometric interpretation of complex numbers* Argand did not include his name in title page. After several years it reached the journal Annals of Mathematics which published it in 1813 mentioning the missing authorship and urging the unknown author to turn up. Argand came to know this and replied to the journal.

It came into wide acceptance with the work of Gauss in 1831.

A complex number $z = x + iy$ is represented by the point (x, y) in the plane.
Distance between two points $z = (x, y)$ and $w = (a, b)$ is

$$|z - w| = \sqrt{(x - a)^2 + (y - b)^2}.$$

$$|z| = \sqrt{x^2 + y^2}.$$

A circle with centre w and radius r is given by

$$|z - w| = r.$$

The straight line passing through $w = (a, b)$ and $w' = (c, d)$ has equation

$$z = (1 - t)w + tw'$$

where $t \in \mathbb{R}$. Or

$$z = w + t(w' - w)$$

where $t \in \mathbb{R}$.
This line passes through w and has direction $w' - w$.

Polar Representation

Uses magnitude and direction of $z = (x, y)$ to represent it.
Magnitude is $|z|$.
Direction is given by the angle the line from origin to (x, y) makes with the positive real axis.

Thus $z = (1, 0)$ has representation $(1, 0)$.
$(1, 1)$ has representation $(\sqrt{2}, \pi/4)$
$(0, 1)$ has representation $(1, \pi/2)$
$(-1, -1)$ has representation $(\sqrt{2}, \pi + \frac{\pi}{4})$

Qn. Which is the complex number whose polar representation is $(1, 3\pi/2)$.

Qn. Which is the complex number whose polar representation is $(1, 3\pi/4)$.

Let $z = (1, \theta)$ be the polar representation of a complex number z. Then geometrically we can see that in Cartesian coordinates

$$z = (\cos\theta, \sin\theta).$$

That is

$$z = \cos\theta + i\sin\theta.$$

Multiplication

With exponential notation we can write

$$z = e^{i\theta}$$

so that

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Note that

$$|e^{i\theta}|^2 = |\cos\theta + i\sin\theta|^2 = \cos^2\theta + \sin^2\theta = 1.$$

Multiplication has a simple realization as follows. For $z = re^{i\theta}$ and $w = se^{i\phi}$

$$zw = rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi$$

$$= rs(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi))$$

That is

$$zw = rs(\cos(\theta + \phi)) + i\sin(\theta + \phi) = rse^{i(\theta+\phi)}.$$

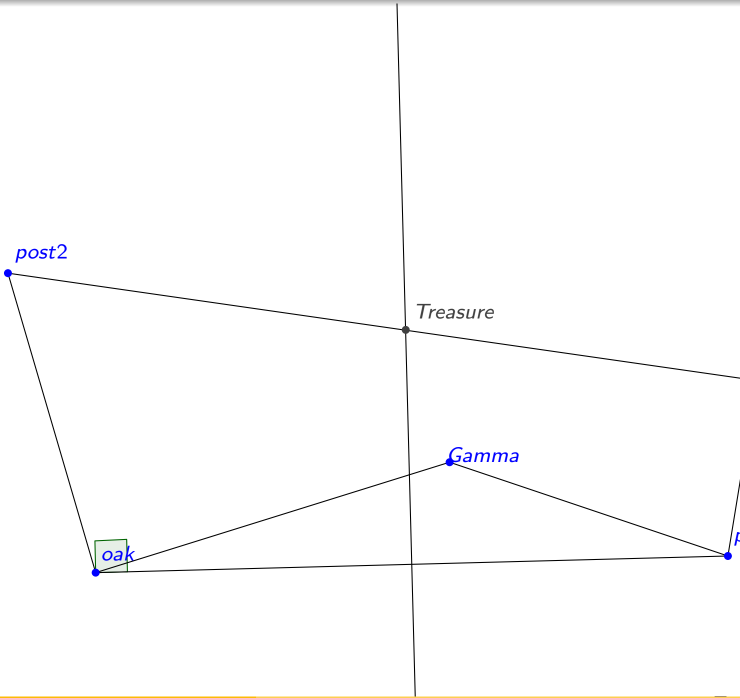
Since $i = e^{i\frac{\pi}{2}}$ we see that

$$iz = re^{i(\theta + \frac{\pi}{2})}.$$

That is multiplication by i amounts to rotation by an angle $\pi/2$.

With this much geometry of the complex plane we attempt to solve the Treasure Hunt of Gamov.

Fix a position Γ as the Hanger tree and follow steps.


post2
Treasure
Gamma
oak

The post positions are

$$(\Gamma - 1)(-i) + 1$$

and

$$(\Gamma + 1)i - 1$$

So mid point is

$$\frac{\text{sum}}{2} = i.$$

Computations

Qn. Is $(1+i)^n$ real for some n.
Find $(1+i)^n$.

Use polar form.

$$1 + i = \sqrt{2}e^{i\pi/4}.$$

So

$$(1 + i)^n = (\sqrt{2})^n e^{in\pi/4}$$

Qn. Is $(1+i)^n + (1-i)^n = 0$ for some n.

Qn. Is $(1+i)^n - (1-i)^n = 0$ for some n.

nth Roots

Qn. Find all cube roots of 1.

$$1 = e^{i2\pi}$$

So one of the cube roots is

$$e^{i2\pi/3}.$$

But

$$1 = e^{i2\pi} = e^{i4\pi} = e^{i6\pi} = e^{i8\pi} = \cdots$$

So

$$e^{i2\pi/3},\ e^{i4\pi/3},\ e^{i6\pi/3},\ e^{i8\pi/3}\ldots$$

are also roots.

Qn. Are they all the roots.

Qn. Find all 5th roots of 1.

$$1 = e^{i2\pi}$$

So one fifth root is

$$e^{i2\pi/5}.$$

But

$$1 = e^{i2\pi} = e^{i4\pi} = e^{i6\pi} = e^{i8\pi} = \cdots$$

So

$$e^{i2\pi/5},\ e^{i4\pi/5},\ e^{i6\pi/5},\ e^{i8\pi/5} \ldots$$

are also roots.

Qn. Are they all the roots.

Qn. Find all fourth roots of -1.

Qn. Find all sqare roots of i.

Qn. Find all sqare roots of $-i$.

Qn. Factorize

$$x^4 + 1 \text{ in } \mathbb{R}[x].$$

$$(x^2 + i)(x^2 - i).$$

This is in $\mathbb{C}[x]$

The fourth roots of -1 are

$$\alpha = \frac{1+i}{\sqrt{2}},\ \beta = -\frac{1+i}{\sqrt{2}},\ \frac{1-i}{\sqrt{2}},\ -\frac{1-i}{\sqrt{2}}.$$

Taking $(x-\alpha)(x-\bar{\alpha})$ and $(x-\beta)(x-\bar{\beta})$ we get

$$x^4+1=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1).$$

Number Theory

Solution of real quadratic equations leads to complex numbers. So solution of quadratics with complex coefficients may lead to a new system of numbers.

D'Alembert conjectured that complex numbers will suffice.

Gauss in 1799 established the **Fundamental Theorem of Algebra**.

Example 1

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

Consider $e^{i\theta}$.

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{i^3\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots$$

Now

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

and

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

It follows

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

log

Let $z = re^{i\theta} = |z|e^{i\theta}$.

$$\log z = \log r + i\theta$$

Some Anomalies What is $\log i$.

Leibnitz argued as follows

$$\begin{aligned} \log i &= \log(-1)^{\frac{1}{2}} \\ &= \frac{1}{2}\log(-1) \end{aligned}$$

And

$$\begin{aligned} 2\log(-1) &= \log(-1)^2 \\ &= \log 1 = 0 \text{ giving} \\ \log(-1) &= 0. \end{aligned}$$

Therefore

$$\log i = \frac{1}{2}\log(-1) = 0.$$

But $e^0 = 1$.

Bernouli

argued differently.

Describe

$$e^i \quad \text{or more generally } e^z.$$

From Euler's relation

$$e^{i\frac{\pi}{2}} = \cos(\pi/2) + i\sin(\pi/2) = i.$$

So

$$\log i = \frac{\pi}{2}i.$$

Similar fallacies can occur in dealing with square roots also.
For example

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

is valid for positive reals.
If it is applied to negative reals we get

$$\begin{aligned} 1 &= \sqrt{1} \\ &= \sqrt{(-1)(-1)} \\ &= \sqrt{(-1)}\sqrt{(-1)} \\ &= i \cdot i \\ &= -1. \end{aligned}$$

Limits and Complex functions

Find the limit of

$$f(z) = \frac{\bar{z}}{z} = \frac{x - iy}{x + iy}$$

at $z = 0$.

If $z \longrightarrow 0$ along $x = y$ then

$$\lim_{z \to 0} f(z) = \frac{1 - i}{1 + i}.$$

If $z \longrightarrow 0$ along $x - axis$ then $y = 0$ and so

$$\lim_{z \to 0} f(z) = 1.$$

If $z \longrightarrow 0$ along $y - axis$ then $x = 0$ and so

$$\lim_{z \to 0} f(z) = -1.$$

Topology of the complex plane.

Domains of funcions are geneally regions.

These are connected open sets.

Power Series

$$\sum a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

For example the geometric series

$$1 + z + z^2 + \cdots$$

Here the partial sum is

$$1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z}.$$

The series converges to $\frac{1}{1-z}$ for $|z| < 1$
and diverges for $|z| > 1$.

Radius of convergence

Theorem 2

For each power series $\sum a_n z^n$ *there exists* R *with* $0 \leq R \leq \infty$ *such that*

1. *the series converges for* $|z| < R$ *and the limit function is analytic.*
2. *For* $|z| > R$ *the series is divergent.*
3. *the series is absolutely convergent for* $|z| < R$ *and for* $0 \leq \rho < R$ *the convergence is uniform for* $|z| \leq \rho$*.*
4. *When* $|z| < R$ *the derivative of the limit function is the series obtained by termwise differentiation and the derived series has the same radius of convergence* R*.*

Formula for radius of convergence

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$

Also

$$\lim \frac{|a_n|}{|a_{n+1}|} = R$$

whenever the limit exists.

Extended Plane : Riemann Sphere

In the real case we sometimes consider the extended real number system by adjining the points $+\infty$ and $-\infty$.
In the complex case extended sysytem is obtained by adjoining one point at infinity such that all straight lines when extended passes through this point.

A geometric model of the extended complex plane is known as Riemann Sphere

Let S denote the sphere given by

$$x_1^2 + x_2^2 + x_3^2 = 1.$$

With every point (x_1, x_2, x_3) on S except $(0, 0, 1)$ we associate the complex number

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

The above correspondence is one to one and given a complex number z we can get the corresponding point (x_1, x_2, x_3) as follows.

$$x_1 = \frac{z + \bar{z}}{1 + |z|^2}, \; x_2 = \frac{z - \bar{z}}{i(1 + |z|^2)}$$

and

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

The point $(0, 0, 1)$ is taken as the point at infinity.

Observations

- When $x_3 = 0$ we have $z = x_1 + ix_2$. That means all points of the sphere lying on the $x - y$ plane are mapped to itself.
- The point $(0, 0, -1)$ on S corresponds to $z = 0$.
- The point $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$ on S corresponds to $z = \frac{\sqrt{3}+i\sqrt{3}}{1+\sqrt{3}}$ so that $|z| < 1$.
- The point $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ on S corresponds to $z = \frac{\sqrt{3}+i\sqrt{3}}{1-\sqrt{3}}$ so that $|z| > 1$.

- Any point (x_1, x_2, x_3) on S with $x_3 > 0$ corresponds to z such that $|z| > 1$ and
- Any point (x_1, x_2, x_3) on S with $x_3 < 0$ corresponds to z such that $|z| < 1$.

Stereographic Projection

The correspondece from points of sphere to the points of the complex plane can be viewed geometrically as a projection.

The points $(0, 01)$, (x_1, x_2, x_3) on S and the corresponding z all lie in a straight line.

So z can be realised as the point on the plane meeting the line joining $(0, 01)$ and (x_1, x_2, x_3).

This projection is called *Stereographic Projection*

The metric in the Riemann sphere is the usual distance in the 3−space.

If z denotes the point on the Riemann sphere corresponding to the complex number z then

$$d(z,0) = \frac{2|z|}{\sqrt{1+|z|^2}}$$

and

$$d(z,\infty) = \frac{2}{\sqrt{1+|z|^2}}.$$

Note that 0 on the Riemann sphere is the point $(0,0,-1)$ and ∞ is the point $(0,0,1)$.

Integration

Theorem 3

If $f(z)$ is analytic in a region bounded by a closed curve γ and if f has no singularity on the curve then

$$\int_{\gamma} f(z)\,dz = 0.$$

Liouville's theorem that

every bounded analytic function is a constant

makes anlytic function theory dinstinctive from real function theory.

Lagrange 1797: A function is analytic if it has a power series expansion

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Cauchy 1814: A function is analytic if it is continuous and satisfy the Cauchy-Riemann equations

$$u_x = v_y; \ u_y = -v_x$$

for $f = u + iv$.

Weierstrass 1860: A function is analytic if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists in the region considered.

- Real valued analytic functions are constants.
- Purely imaginery valued analytic functions are also constants.
- Analytic functiions with constant modulus are also constants.
- Real and imaginery parts are connected by Cauchy Riemann equations

$$u_x = v_y; \; u_y = -v_x$$

where the function is $u + iv$.

- Real and imaginery parts are Harmonic functions. That is contunuous real valued functions $u(x, y)$ satisfying

$$u_{xx} + u_{yy} = 0.$$

Theorem 4 (Cauchy's Integral Formula)

Let $f(z)$ be analytic in a region bounded by a simple closed curve γ. Then or any a in the region

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a}\, dz$$

and the nth derivative

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz$$

Singularities

Removable Singularity: $f(z)$ is said to have a removable singularity at a point $z = a$ if

$$\lim_{z \to a} (z - a) f(z) = 0.$$

Pole : $f(z)$ is said to have a Pole at a point $z = a$ if

$$\lim_{z \to a} f(z) = \infty$$

and in this case

$$\lim_{z \to a} (z - a)^n f(z) = 0 \text{ for some positive integer } n.$$

Essential singularity: $f(z)$ is said to have an essential singularity at a point $z = a$ if $\lim_{z \to a}$ does not exist. In this case

$$\lim_{z \to a} (z - a)^n f(z) \text{ is never zero.}$$

Laurent Series

A series of the form

$$\sum_{-\infty}^{+\infty} a_n(z-a)^n$$

is called the Laurent series about the point $z = a$.

- If $z = a$ is a Removable singularity of $f(z)$ then $a_n = 0$ for all $n < 0$. In this case the limit function is analytic.
- If $z = a$ is a Pole of $f(z)$ then $a_n = 0$ for all $n < k$ for some negative integer k.
- If $z = a$ is a n Essential singularity of $f(z)$ then $a_n \neq 0$ for infinitely many $n < 0$.

Compare the singularities of

$$\frac{e^z - 1}{z}, \ \frac{1}{z^2} \text{ and } e^{1/z}$$

All these have singularities at $z = 0$.

Note the difference.

$$\lim_{x\to 0} e^{1/x} = \infty.$$

But

$$\lim_{z\to 0} e^{1/z} \neq \infty.$$

This limit does not exist.

For if $z \to 0$ along $\frac{i}{2n\pi}$ then

$$\lim_{z\to 0} e^{1/z} = \lim_{n\to\infty} e^{-2n\pi i} = 1.$$

So $z = 0$ is not a pole of the function.

Residue

If $f(z)$ has a pole at $z = a$ then the coefficient of $(z-a)^{-1}$ in the Laurent series expansion of $f(z)$ around a is the *Residue* of $f(z)$ at $z = a$.

- If a is a simple pole then Residue at a is

$$\lim_{z \to a} (z-a) f(z)$$

- If a is a simple pole of order n then Residue at a is

$$\lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

Example 5

1. $f(z) = \frac{\sin z}{z^3}$ *has a pole of order* 2 *at* $z = 0$.

Residue at $z = 0$ *is*

$$\lim_{z \to 0} \frac{d}{dz}\left(z^2 \frac{\sin z}{z^3}\right) = \lim \frac{z \cos z - \sin z}{z^2}.$$

We we consider the Laurent expansion

$$\frac{\sin z}{z^3} = \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots$$

So Residue $=$ *coefficient of* $\frac{1}{z} = 0$.

Example 6

1. $f(z) = \frac{z}{(z+1)^2}$ *has a pole of order* 2 *at* $z = -1$.
Residue at $z = -1$ *is*

$$\lim_{z \to -1} \frac{d}{dz}((z+1)^2 f(z)) = 1$$

2. $f(z) = \frac{z^2}{(z-1)^2}$ *has a pole of order* 2 *at* $z = 1$.
Residue is

$$\lim_{z \to 1} \frac{d}{dz}((z-1)^2 f(z)) = 2.$$

Theorem 7 (Cauchy's Residue Theorem)

Let $f(z)$ be analytic in a region bounded by a simple closed curve γ except for some isolated singularities $a_1, a_2, \ldots, a_k$ none of which are on the curve. Then

$$\frac{1}{2\pi i}\int_{\gamma} f(z) = \sum_{j=1}^{k} R_j$$

where R_j is the residue of f at a_j.

Applications

Evaluate

$$\int_0^\infty \frac{\sin x}{x}\,dx.$$

Evaluation is simplified by applying complex integration.

Evaluate

$$\int_{\gamma_R} \frac{\sin z}{z}\,dz$$

where γ_R is a semicircle of radius R centered at origin.

$$\int_{\gamma_R} \frac{\sin z}{z} dz = 0$$

since residue at 0 is 0.

It follows that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = 0$$

and so

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = 0.$$

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