IMRT WORKSHOP ON FOUNDATIONS OF ABSTRACT ANALYSIS

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Session 1. Convergence and Continuity

Consider the following questions

1. Are there integer solutions to the equation

$$2x + 3y = 4$$

2.

Are there integer solutions to the equation

$$2x + 6y = 7$$

3. Are there integer solutions to the polynomial equation

$$2x^5 + 3x^4 + 6x + 1 = 0$$

A. Can the sum of infinitely many reals be a finite real.

B. Consider a sum of infinitely many rationals. Is it always a rational.

Consider the following computations.

$$\begin{aligned} S &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\ 2S &= 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots \\ &= 2 + S \text{ so that} \\ S &= 2. \end{aligned}$$

Compare it with the following.

$$\begin{aligned}
\Sigma &= 1+2+4+8+\cdots \\
2\Sigma &= 2+4+8+\cdots \\
&= -1+1+2+4+8+\cdots = -1+\Sigma \text{ so that} \\
\Sigma &= -1
\end{aligned}$$

Obviously unacceptable.

Real Number System

The set $\mathbb{R}$ of all real numbers has the following properties

- Algebraically a field.
- Totally ordered and ordered field.
- Archimedean Property. That is, given positive reals a, b there is a positive integer n such that $na > b$.
- Completenes with respect to the order.

Completeness

Completeness of the set of all real numbers is the most basic property on which the analysis of real numbers depends.

This property asserts the existence of a unique Least Upper Bound (LUB) for all subsets which are bounded above.

Also it asserts the existence of a unique Greatest Lower Bound (GLB) for all subsets which are bounded below.

Example 1

The set $A = \{x : x^2 + x + 1 < 7\}$ *is bounded above.*

Every element of A *is less than* 2.

Also in this case 2 *is the LUB of* A.

Qn. *Is* $2 \in A$.

As another example consider

Example 2

The set $B = \{x : x^2 - x < 1\}$ *is bounded above.*

Every element of A *is less than* 2.

In this case the LUB of B *is not that visible. But by completeness property we have a LUB say* β.

Qn. *Is* $\beta \in B$.

Qn. *Is* B *bounded below.*

Observe that LUB and GLB may not exist in the set $\mathbb{Q}$ of rationals.
The set $B = \{x \in \mathbb{Q} : x^2 - x \leq 1\}$ is bounded above.
Every element of A is less than 2.
In this case the LUB of B does not exist if our domain of activity is set of rationals only.

Theorem 3

(i) In every interval (a, b) of the real line there is a rational number.
(ii) In every interval (a, b) of the real line there is an irrational number.

Series and Sequences

Decimal expansion gives

$$\frac{1}{3} = \cdot 33333 \cdots = \cdot 3 + \cdot 03 + \cdot 003 + \cdots$$

This is an infinte sum, a SERIES.

Conversely consider the series

$$\cdot 3 + \cdot 03 + \cdot 003 + \cdots$$

and find the sum.

Example 4

Construct a sequence of rationals converging to $\sqrt{2}$.

$$a_1 = 1.4,\ a_2 \in \mathbb{Q} \cap (\frac{a_1 + \sqrt{2}}{2}, \sqrt{2}),\ a_3 \in \mathbb{Q} \cap (\frac{a_2 + \sqrt{2}}{2}, \sqrt{2}), \ldots$$

Then $\sqrt{2} - a_n$ can be made arbitrarily small for large enough n.
We use the fact that every interval contains a rational number.

Another well known approximating sequence for $\sqrt{2}$ is given by the Babylonian method of approximation to $\sqrt{2}$.

$$a_1 = 1 \text{ (any choice)}; a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}).$$

This gives

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{3}{2} \\ a_3 &= \frac{17}{12} \\ a_4 &= \frac{577}{408} \\ \vdots &= \vdots \end{aligned}$$

Definition

A sequence (x_n) of real numbers is said to converge to a real number L if the following hold.
Given $\epsilon > 0$ there exists a natural number N such that

$$|x_n - L| < \epsilon \text{ for all } n \geq N.$$

Test the convergence of the sequence $(\frac{1}{n})$.

Test whether $(\frac{1}{n})$ converges to 1.

Test whether $(\frac{1}{n})$ converges to $1/2$.

Test whether $(\frac{1}{n})$ converges to $1/4$.

Test whether the sequence $(1/n)$ converges to zero. Given $\epsilon = 1/2$ we can choose $N = 3$. Then $a_n < \epsilon$ for all $n \geq 3$.
Given $\epsilon = 1/4$ we can choose $N = 5$. Then $a_n < \epsilon$ for all $n \geq 5$.
Given $\epsilon = 3/100$ we can choose $N = 100$. Then $a_n < \epsilon$ for all $n \geq 100$ etc.

Example 5

Show that the sequence $(1/\sqrt{n})$ *converges to zero. Given* $\epsilon = 1/2$ *we can choose* $N = 5$. *Then* $a_n < \epsilon$ *for all* $n \geq 5$
Given $\epsilon = 1/4$ *we can choose* $N = 17$. *Then* $a_n < \epsilon$ *for all* $n \geq 17$
Given $\epsilon = 1/10$ *we can choose* $N = 101$. *Then* $a_n < \epsilon$ *for all* $n \geq 101$ *etc.*

Limit of functions

Definition Let f be a real valued function defined on an interval $[a, b]$. L is said to be the limit of $f(x)$ at $x = c$ if

$$\text{for every } \epsilon > 0 \text{ there exists } \delta > 0 \text{ such that}$$

$$|f(x) - L| < \epsilon \text{ whenever } |x - c| < \delta.$$

Continuous Functions

Usual classes of functions

- Polynomials
- Rational functions
- Trigonometric functions
- Exponential functions
- Rational functions
- Greatest integer function

Example 6

Discuss the continuity of $f(x) = \frac{x^2-1}{x-1}$ *at* $x = 1$.

Observe that $f(x)$ *is not defined at* $x = 1$.

But we may find whether $f(x)$ *has a limit at* $x = 1$.

See some of the values

$$\begin{aligned} f(0) &= 1 \\ f(1/2) &= 1.5 \\ f(3/4) &= 1.75 \\ f(7/8) &= 1.875 \text{ etc.} \end{aligned}$$

We may show that this limit is 2.

Intermediate value property

Determination of zeros of $f(x) = e^x - 5x$.

$f(0) = 1$ and $f(1),\ f(2)$ are negative and $f(3)$ is positive.

So there is one zero in $(0, 1)$ and one zero in $(2, 3)$.

Uniform Convergence

Example 7

Let $f_n(x) = x^n$ *for* $x \in [0, 1]$. *Then*

$$f_n(x) \text{ converges to } 0 \text{ for } x < 1$$

and

$$f_n(x) \text{ converges to } 1 \text{ for } x = 1.$$

Note that each $f_n(x)$ *is continuous. But* $f(x) = \lim f_n(x)$ *is not continuous at* $x = 1$.

Theorem 8

If a sequence $(f_n(x))$ of continuous functions congverges uniformly in an interval then the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ is also continuous.

Example 9

Let $f_n(x) = x^n$ *for* $x \in [0, \frac{1}{2}]$*. Then*

$$f_n(x) \text{ converges to } 0 \text{ for } x$$

so that $f(x) = \lim f_n(x) = 0$ *for all* x.
It can be seen that $(f_n(x))$ *converges uniformly in* $[0, \frac{1}{2}]$*. Further the limit function* $f(x)$ *is continuous in* $[0, \frac{1}{2}]$.

Now consider $f_n(x) = x^n$ for x in the open interval $(0, 1)$. Then

$$f_n(x) \text{ converges to } 0 \text{ for } x$$

so that $f(x) = \lim f_n(x) = 0$ for all x.
Here the limit function is continuous but the convergence is not uniform.

Example 10

$f_n(x) = \frac{\sin nx}{n!}$ *is uniformly convergent in* $\mathbb{R}$.

The series $\sum \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ *is uniformly convergent in any interval.*

[1] Euler, Analysis of the Infinite.

[2] Robert Rogers and Eugene Boman, A Story of Real Analysis, Open book, 2014.

[3] Tao Terrence, Real Analysis.

[4] Toeplitz, The Calculus. A Genetic Approach, 1930 (German), 1963(English).

IMRT Analysis Workshop
Questions

1. Describe a collection of infinitely many rationals whose sum is a rational number.
2. Describe a collection of infinitely many rationals whose sum is an irrational number.

3. Find a choice of N corresponding to the following given ϵ in determining the convergence of the sequence $(1/n^2)$.
(i) $\epsilon = 1$, (ii) $\epsilon = 1/2$ (iii) $\epsilon = 1/10$.

4. Find $\lim_{n\to\infty} \sqrt[n]{2}$.

5.Verify whether the sequence $(\sqrt[n]{n})$ is convergent. If so find the limit.

1. Show that $\sqrt[n]{n} > \sqrt[n+1]{n+1}$ for $n \geq 3$.
2. Show that $(\sqrt[n]{n}) \geq 1$ for all n.

6. Show that $(1 - \frac{1}{n})^{-n}$ converges to e. Use the fact that $(1 + \frac{1}{n})^{n}$ converges to e.

7. Verify the convergence of the following series

1. $\sum \frac{1}{n}$
2. $\sum \frac{1}{\sqrt{n}}$
3. $\sum \frac{a^n}{\sqrt{n!}}$
4. $\sum \frac{(-1)^n}{2n+1}$ (Madhava–Grigory Series for $\pi/4$)

8. Discuss the meaning of $2^{\sqrt{2}}$ in the form that
$2^n = 2 \times 2 \times 2 \times \cdots \times 2$ (n times).

9. Describe the continuity of $f(x) = x\sin(1/x)$.

10. Describe the continuity of $f(x) = [x]$ the greatest integer less than or equal to x.

11. Describe the continuity of $f(x) = \begin{cases} 1 + x \text{ if } x \leq 1 \\ 2 \text{ otherwise.} \end{cases}$

12 Describe the continuity of $f(x) = \begin{cases} \frac{1+x}{1-x} \text{ if } x \neq 1 \\ 2 \text{ otherwise.} \end{cases}$