Introduction to Complex Functions

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November 16, 2018

What is a real function?

IMRT

Elementary functions
Square Roots
Argument
Branches
Trigonometric
Exponential
Logarithm
Power
Analyticity
Power Series
vs Real
vs Polynomials
Problems

Elementary functions

What is a real function?

Operations done on real numbers to produce other real numbers

Elementary functions

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- $f(x) = \dfrac{x^3 - 3x + 2}{x^2 + 1}$

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- $f(x) = \sqrt{x}$

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What about square roots?

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The first two continuous in $[0, \infty)$

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$$z = r(\cos\theta + \mathrm{i}\sin\theta)$$

if we define

$$w = \sqrt{r}(\cos\tfrac{1}{2}\theta + \mathrm{i}\sin\tfrac{1}{2}\theta)$$

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θ not uniquely determined by z

Argument function

For each non-zero complex number, we can choose θ such that

$$z = |z|\,(\cos\theta + \mathrm{i}\sin\theta) \quad \text{and} \quad -\pi < \theta \leq \pi$$

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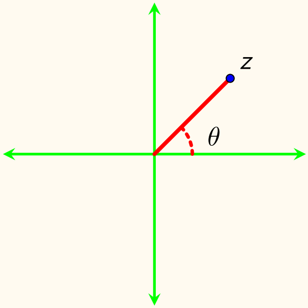

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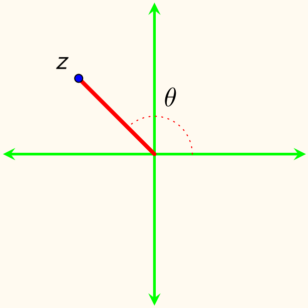

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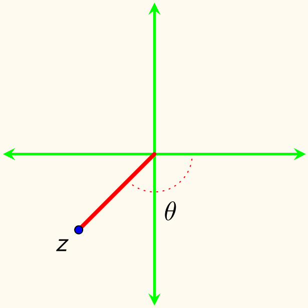

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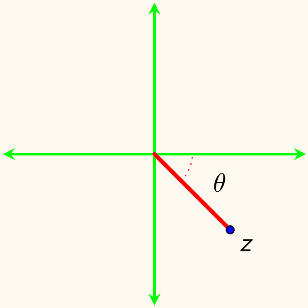

Argument function

For $z = x + \mathrm{i}y$, define

$$\theta = \begin{cases} \cos^{-1} \frac{x}{r}, & \text{if } y \geq 0 \\ -\cos^{-1} \frac{x}{r}, & \text{if } y \leq 0 \end{cases}$$

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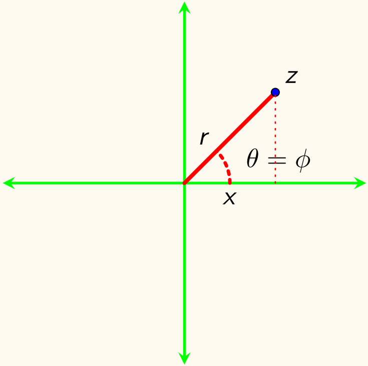

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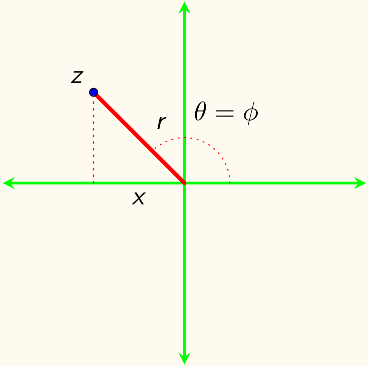

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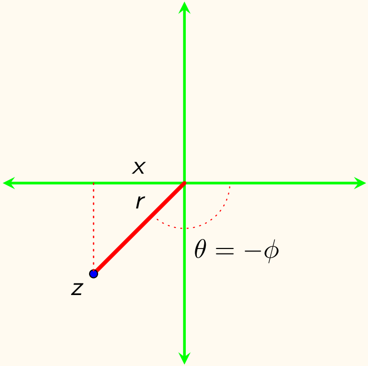

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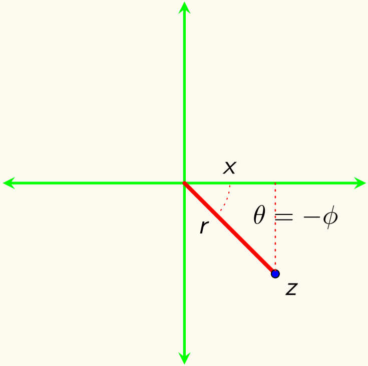

Argument function

For each non-zero complex number z, define

$$\theta(z) = \begin{cases} \cos^{-1} \dfrac{\mathrm{Re}(z)}{|z|}, & \text{if } \mathrm{Im}(z) \geq 0 \\ -\cos^{-1} \dfrac{\mathrm{Re}(z)}{|z|}, & \text{if } \mathrm{Im}(z) < 0 \end{cases}$$

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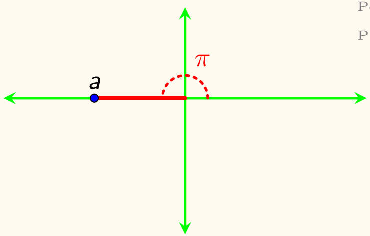

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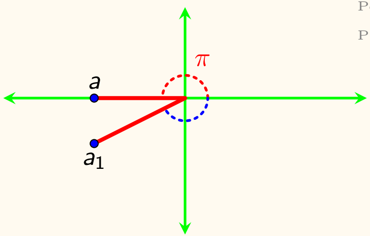

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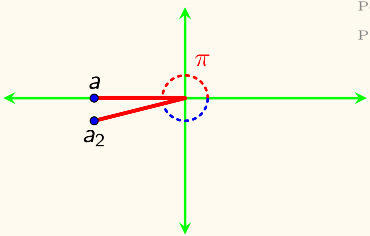

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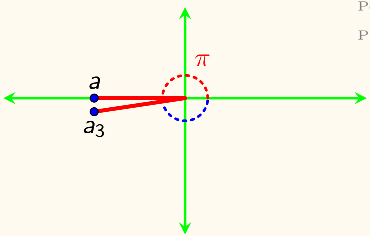

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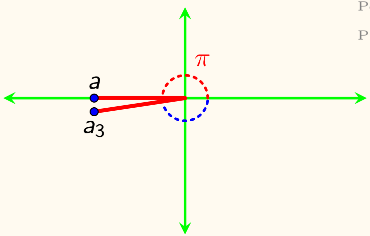

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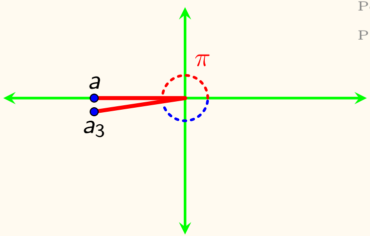

θ continuous on $\mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$

Branches of square root

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Define g on $G = \mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$ by

$$g(z) = \sqrt{|z|}\left(\cos \tfrac{1}{2}\theta(z) + \mathrm{i} \sin \tfrac{1}{2}\theta(z)\right)$$

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$$g(z) = \sqrt{|z|}\left(\cos \tfrac{1}{2}\theta(z) + i \sin \tfrac{1}{2}\theta(z)\right)$$

Then g is continuous in G and $g(z)^2 = z$ for each z in G
Branch of square root function on $\mathbb{C} \setminus \{x \in \mathbb{R} \colon x \leq 0\}$

Problems

- Find the positions of two different square roots of $24 + 7i$ in the complex plane.

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- Find the positions of two different square roots of $24 + 7i$ in the complex plane.
- Find the positions of the four different fourth roots of $24 + 7i$

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- Mark the 6th roots of i

Trigonometric functions

sine and cosine as functions on real numbers

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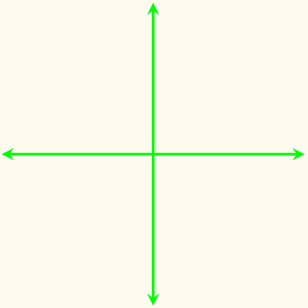

- Cartesian coordinate system

Trigonometric functions

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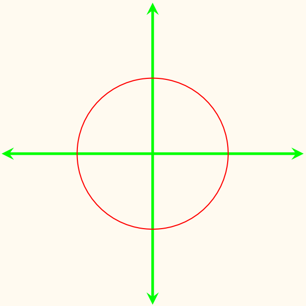

- Cartesian coordinate system
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Trigonometric functions

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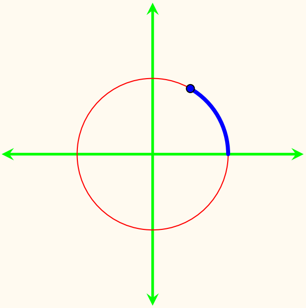

- Cartesian coordinate system
- Circle of radius 1, centered at the origin
- $t \geq 0$. Arc-distance t from $(0, 1)$ counter-clock-wise

Trigonometric functions

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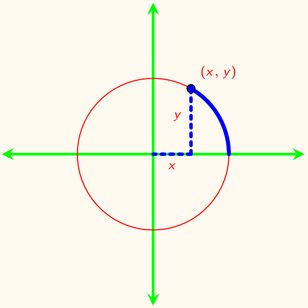

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- Coordinates of this point

Trigonometric functions

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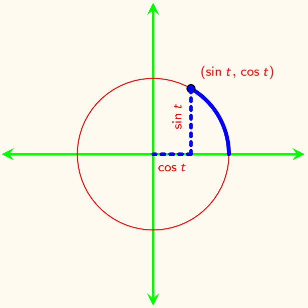

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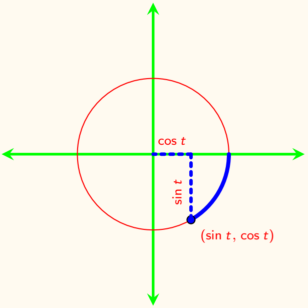

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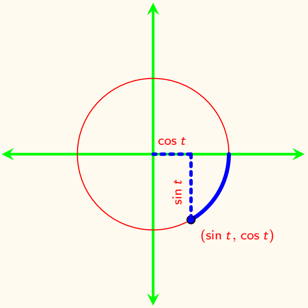

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- $t < 0$. Clockwise arc
- Graph

Trigonometric functions

Madhavan series for real sine and cosine functions

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

$$\cos x = 1 - \frac{1}{1!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

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These series converge for complex numbers also

We *define* for every complex number z

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Exponential function

The series

$$1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

converges for every complex number z

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We define the function exp on $\mathbb{C}$ by

$$\exp(z) = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

Exponential function

The series

$$1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

converges for every complex number z

We define the function exp on $\mathbb{C}$ by

$$\exp(z) = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

We can prove that

$$\exp(z)\exp(w) = \exp(z + w) \text{ for all } z, w \text{ in } \mathbb{C}$$

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$$\mathrm{e}^{\mathrm{i}z} = \exp(\mathrm{i}z) = \cos z + \mathrm{i}\sin z \text{ for all } z \text{ in } \mathbb{C}$$

Problems

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- Find the value of $\cos(\mathrm{i})$ (Grater than 1!)
- Check the identity $\cos^2\theta + \sin^2\theta = 1$ for $\theta = \mathrm{i}$

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2. Can you show that $\frac{1}{2}$ may not be equal to $(2^{\mathrm{i}})^{\mathrm{i}}$?

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  of complex numbers may or may not converge
- If it converges to the complex number w, we write

$$w = \sum_{n=0}^{\infty} a_n (z-c)^n$$

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- f defined by the power series

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    - $f(z) = \sum_{n=0}^{\infty} a_n(z-c)^n$ defined only for $z = c$
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In the last two cases, f analytic

Power series

Converse:

If f is analytic in an open ball $B(c;r)$, then there exists a squence $a_0, a_1, a_2, \ldots$ of complex numbers such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

for all z in $B(c;r)$

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Analytic function—infinite analogue of polynomials

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A real differentiable function on an open interval may be extended in infinitely many ways

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  - The set of all real polynomial functions on an interval of real numbers

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- Prove that the function $f(z) = |z|^2$ is differentiable only at 0
- Find the principal value of i^{i}
- Which of the following sets of functions are integral domains with respect to point-wise addition and multiplication:
  - The set of all real polynomial functions on an interval of real numbers
  - The set of all real differentiable functions on an open interval of real numbers

Problems

- Prove that the function $f(z) = \overline{z}$ is not differentiable at any point
- Prove that the function $f(z) = |z|^2$ is differentiable only at 0
- Find the principal value of i^{i}
- Which of the following sets of functions are integral domains with respect to point-wise addition and multiplication:
  - The set of all real polynomial functions on an interval of real numbers
  - The set of all real differentiable functions on an open interval of real numbers
  - The set of all analytic functions on an open connceted set of complex numbers